Long-Range Integrable Spin Chains

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Goal

We want to construct the most general long-range integrable spin chain transforming under an arbitrary symmetry algebra $\mathfrak{g}$ by deformations of an ordinary short-range (e.g. nearest-neighbor) chain [1, 2]. For applications to supersymmetric gauge theory cf. [3] and related works. Start with a $\mathfrak{g}$-invariant short-range spin chain whose integrability is expressed by an infinite tower of conserved charge operators

$$[\mathcal{Q}_n^{(0)}, \mathcal{Q}_m^{(0)}] = 0.$$

We focus on charge operators which act *locally* and *homogeneously* on the spin chain.

Generating Equation

The basic idea is to deform the vector of integrable charges $\mathcal{Q} = (\mathcal{Q}_2, \mathcal{Q}_3, \dots)$ by *parallel transport* :

$$\mathcal{D}\mathcal{Q}(\xi) = 0, \quad \text{with} \quad \mathcal{D} := d - \Xi. \qquad (1)$$

Here $\Xi(\xi)$ denotes a connection to be specified. This deformation implies that the algebra among the charge operators does not change

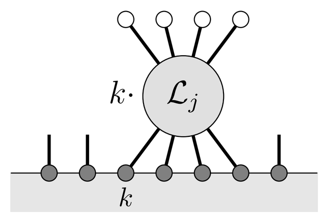
$$[\mathcal{Q}_n, \mathcal{Q}_m] = f_{nmk}\mathcal{Q}_k, \qquad df_{nmk} = 0.$$

In particular, integrability of commuting charges ($f_{nmk} = 0$) is preserved by parallel transport

$$[\mathcal{Q}_n(\xi), \mathcal{Q}_m(\xi)] = 0.$$

That is, taking a set of commuting short-range charges $\mathcal{Q}_n(0) = \mathcal{Q}_n^{(0)}$, one obtains a set of deformed charges $\mathcal{Q}_n(\xi)$ being manifestly integrable.
The connection Ξ has to be chosen such that the deformed charges are local and homogeneous operators.

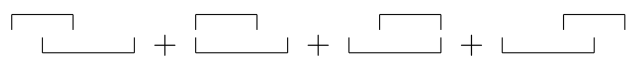
Connection Operators



Boost Charges

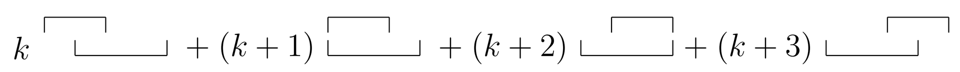
Boost charges $\mathcal{B}[\mathcal{Q}_n]$ are inhomogeneous versions of the homogeneous integrable charges. The action of the homogeneous operator is simply multiplied by its position k on the spin chain. The commutator of a boost charge $\mathcal{B}[\mathcal{Q}_n]$ with a charge $\mathcal{Q}_m$ yields a homogeneous operator independent of k:

$$[\mathcal{Q}_n, \mathcal{Q}_m] =$$



$$= 0$$

$$[\mathcal{B}[\mathcal{Q}_n], \mathcal{Q}_m] =$$



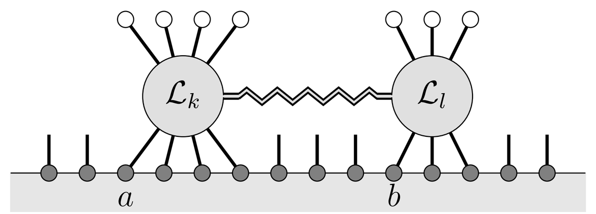
$$= \text{hom}$$

Boost operators modify the one-magnon charge eigenvalues

$$\mathcal{Q}_n|u\rangle = q_n(u \pm \tfrac{i}{2})|u\rangle \to q_n(x(u \pm \tfrac{i}{2}))|u\rangle$$

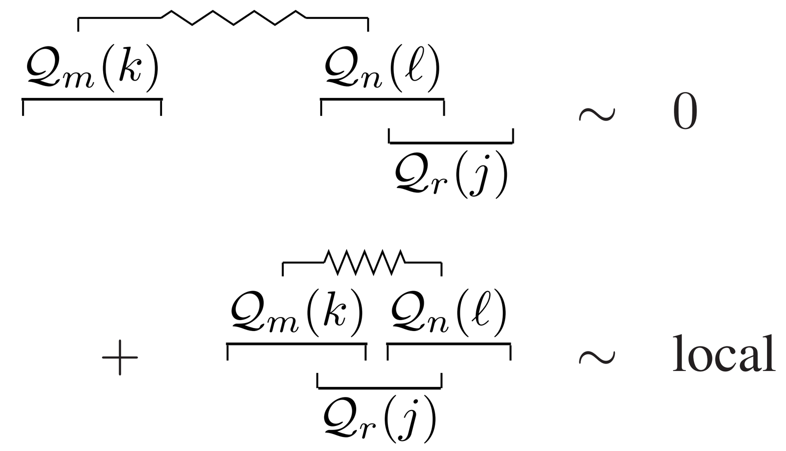
by deformation of the rapidity in form of the *rapidity map* $x(u)$.

Bilocal Charges



Bilocal charges $[\mathcal{Q}_m|\mathcal{Q}_n]$ are compositions of two local charge operators acting on two different positions of the spin chain. The commutator of a bilocal charge with a local charge yields a local operator. Both parts of the bilocal structure commute with the local charge as long as they are well separated from each other:

$$[[\mathcal{Q}_m|\mathcal{Q}_n], \mathcal{Q}_r] =$$

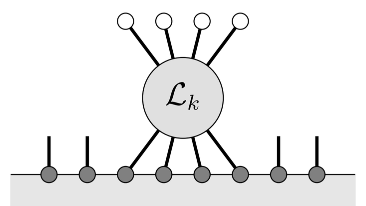


Bilocal charges modify the two-magnon scattering matrix $S = S(u, v)$ by the *dressing phase* $\theta(u, v)$

$$|u\rangle + S|v\rangle \to |u\rangle + e^{2i\theta(u,v)}S|v\rangle.$$

They have no impact on the dispersion relation $q_n(u)$.

Local Operators

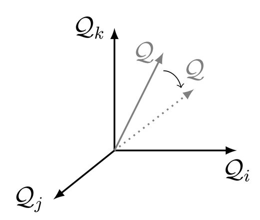


Local operators $\mathcal{L}$ include all local and homogeneous operators invariant under the symmetry algebra $\mathfrak{g}$, i.e. all admissible building blocks for the integrable charges:

$$\mathcal{Q}_n \to e^{i\epsilon\mathcal{L}}\mathcal{Q}_n e^{-i\epsilon\mathcal{L}} = \mathcal{Q}_n^{(0)} + i\epsilon[\mathcal{L}, \mathcal{Q}_n^{(0)}] + \mathcal{O}(\epsilon^2)$$

These similarity transformations have no impact on the spectrum of the charge operators .

Basis of Charges



One can take linear combinations of the charge operators without influence on their algebra:

$$\mathcal{Q}_n \to \mathcal{Q}_n + \gamma\mathcal{Q}_m$$

This degree of freedom can be understood as a generalized rotation of the charge vector $\mathcal{Q} = (\mathcal{Q}_2, \mathcal{Q}_3, \dots)$.

Flatness and Interaction Range

Including the boost, bilocal and rotation connection into (1) we get a covariant derivative of the form

$$\mathcal{D} = d - i\mathcal{B}[\mathcal{Q}_k]\,\Pi_k - i[\mathcal{Q}_r|\mathcal{Q}_s]\,\Upsilon_{r,s} - i\mathfrak{G}_{m,n}\,\Gamma_{m,n}. \qquad (2)$$

Parallel transport with respect to this derivative yields general long-range charges. To reflect gauge/string properties, one has to choose a λ-parameterized path in moduli space $\{\Pi(\lambda), \Upsilon(\lambda), \Gamma(\lambda)\}$ and order the charges' interaction range on the spin chain with respect to the perturbative expansion in this coupling constant:

$$\mathcal{Q}_n(\lambda) = \mathcal{Q}_n^{(0)} + \lambda\mathcal{Q}_n^{(1)} + \lambda^2\mathcal{Q}_n^{(2)} + \mathcal{O}(\lambda^3). \qquad (3)$$

Remarkably this can be achieved by fixing the connection Γ and imposing flatness. The connection (2) is flat ($\mathcal{D}^2 = 0$) if

$$0 = d\Pi_k + \Gamma_{p,k} \wedge \Pi_p\,,$$
$$0 = d\Upsilon_{r,s} + \Gamma_{p,r} \wedge \Upsilon_{p,s} + \Gamma_{p,s} \wedge \Upsilon_{r,p}\,,$$
$$0 = d\Gamma_{m,n} - \Gamma_{m,p} \wedge \Gamma_{p,n}\,.$$

Flatness guarantees an expansion of the form (3) to be independent of the path in moduli space along which one deforms. The four types of connection operators generate the whole long-range moduli space found for the closed $\mathfrak{g} = \mathfrak{gl}(N)$ spin chain [4].

References

[1] T. Bargheer, N. Beisert and F. Loebbert, *"Boosting Nearest-Neighbour to Long-Range Integrable Spin Chains"*, J. Stat. Mech. 0811, L11001 (2008), `arxiv:0807.5081`.

[2] T. Bargheer, N. Beisert and F. Loebbert, *"Long-Range Deformations for Integrable Spin Chains"*, `arxiv:0902.0956`.

[3] N. Beisert, C. Kristjansen and M. Staudacher, *"The Dilatation Operator of $\mathcal{N} = 4$ Conformal Super Yang-Mills Theory"*, Nucl. Phys. B664, 131 (2003), `hep-th/0303060`.

[4] N. Beisert and T. Klose, *"Long-Range $GL(n)$ Integrable Spin Chains and Plane-Wave Matrix Theory"*, J. Stat. Mech. 06, P07006 (2006), `hep-th/0510124`.

Deformed Bethe Equations

The short-range Bethe equations as well as the charge eigenvalues are modified through the deformations induced by (1). While local similarity transformations are compatible with the boundary conditions, rotation, boost and bilocal deformations twist the boundary conditions. The result is a set of long-range asymptotic Bethe equations as they appear in gauge/string dualities. Boost deformations $\mathcal{B}[\mathcal{Q}_n]$ and rotations $\Gamma_{m,n}$ introduce the rapidity map $x(u_k)$ into the Bethe equations. Bilocal deformations $[\mathcal{Q}_n|\mathcal{Q}_m]$ yield the dressing phase $\theta(u_k, u_j)$. The validity of these long-range systems is limited to the asymptotic region where the underlying spin chains do not exceed the interaction range of the operators.

$$\frac{(u_k+\frac{i}{2})^L}{(u_k-\frac{i}{2})^L} = \prod_{\substack{j=1\\j\neq k}}^{M}\frac{u_k-u_j+i}{u_k-u_j-i}$$

$\xrightarrow{\mathcal{B}[\mathcal{Q}],\ \Gamma}$

$$\frac{x(u_k+\frac{i}{2})^L}{x(u_k-\frac{i}{2})^L} = \prod_{\substack{j=1\\j\neq k}}^{M}\frac{u_k-u_j+i}{u_k-u_j-i}$$

$\xrightarrow{[\mathcal{Q}|\mathcal{Q}]}$

$$\frac{(u_k+\frac{i}{2})^L}{(u_k-\frac{i}{2})^L} = \prod_{\substack{j=1\\j\neq k}}^{M}\frac{u_k-u_j+i}{u_k-u_j-i}\,e^{2i\theta(u_k,u_j)}$$

$\longrightarrow$

$$\frac{x(u_k+\frac{i}{2})^L}{x(u_k-\frac{i}{2})^L} = \prod_{\substack{j=1\\j\neq k}}^{M}\frac{u_k-u_j+i}{u_k-u_j-i}\,e^{2i\theta(u_k,u_j)}$$