Bethe equations for generalized Hubbard models

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Abstract

hep-th/09064512

We present the Bethe equations for generalized Hubbard models, based on the coordinate Bethe ansatz. We illustrate the results for various examples corresponding to $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$ algebra. We give some hints how to deal with the general case.

$gl(\mathfrak{n}|\mathfrak{m})$ Hubbard model

- The generalized Hubbard model's R-matrix comes from the coupling of two independent XX models.
- Two XX models can be based on *two different (super)algebras and two different "projectors"* $\pi_\uparrow$ *and* $\pi_\downarrow$ defined below.

$$R_{12}(\lambda) = \Sigma_{12} P_{12} + \Sigma_{12} \sin\lambda + (\mathbb{I}\otimes\mathbb{I} - \Sigma_{12}) P_{12} \cos\lambda \quad (1)$$

where P_{12} is the graded permutation operator and $\Sigma_{12} = \pi_1 \overline{\pi}_2 + \overline{\pi}_1 \pi_2$.

- The *projectors* can be defined for some set of integers $\mathcal{N}$: $\pi = \sum_{j\in\mathcal{N}} E^{jj} \quad , \quad \overline{\pi} = \mathbb{I} - \pi$
- the Hubbard model's R-matrix is

$$R^{Hub}_{\uparrow\downarrow\, 12}(\lambda_1,\lambda_2) = R^{XX}_{\uparrow\, 12}(\lambda^-_{12})\, R^{XX}_{\downarrow\, 12}(\lambda^-_{12}) + \frac{\sin\lambda^-_{12}}{\sin\lambda^+_{12}} \tanh\left(h(\lambda_1)+h(\lambda_2)\right) R^{XX}_{\uparrow\, 12}(\lambda^+_{12})\, C_{\uparrow 1}\, R^{XX}_{\downarrow\, 12}(\lambda^+_{12})\, C_{\downarrow 1}, \quad C_\sigma = \pi_\sigma - \overline{\pi}_\sigma \quad (2)$$

where $\lambda^\pm_{12} = \lambda_1 \pm \lambda_2$ and function $h(\lambda)$ is defined by the same relation as in original Hubbard R-matrix: $\sinh(2h) = U \sin(2\lambda)$.

- The R-matrix (2) satisfies the Yang–Baxter equation.
- The L-site monodromy matrix and its transfer matrix are given by

$$T_{a<b_1\dots b_L>}(\lambda) = R^{Hub}_{\uparrow\downarrow\, ab_1}(\lambda,0)\dots R^{Hub}_{\uparrow\downarrow\, ab_L}(\lambda,0) \;\text{ and }\; t(\lambda) = tr_a T_{a<b_1\dots b_L>}(\lambda) \quad (3)$$

- The generalized Hubbard Hamiltonian with periodic boundary conditions is

$$H = \frac{d}{d\lambda}\ln t(\lambda)\Big|_{\lambda=0} = \sum_{k=1}^{L}\left((\Sigma P)_{\uparrow\, k,k+1} + (\Sigma P)_{\downarrow\, k,k+1} + U\, C_{\uparrow k}\, C_{\downarrow k}\right) \quad (4)$$

Coordinate Bethe Ansatz for $gl(2|1) \oplus gl(2)$ model

As an example we consider a model with 3 different types of "particle": $2\uparrow, 3\uparrow$ and $2\downarrow$ on a vacuum state with the choice of *projectors*: $C_{\uparrow k} = E^{11}_{\uparrow k} - E^{22}_{\uparrow k} - E^{33}_{\uparrow k}$ and $C_{\downarrow k} = E^{11}_{\downarrow k} - E^{22}_{\downarrow k}$

We use the *coordinate Bethe ansatz* method similar to Lieb-Wu:

$$\phi[(\overline{A},\overline{\alpha})] = \sum_{\mathbf{x}} \Psi[\mathbf{x},(\overline{A},\overline{\alpha})] e^{A_1\,\alpha_1}_{x_1} \dots e^{A_N\,\alpha_N}_{x_N}, \quad (5)$$

$$(A,\alpha) = \{(2,\uparrow)\,;\,(3,\uparrow)\,;\,(2,\downarrow)\}$$

with

$$\Psi(\mathbf{x}) = \sum_{P\in\mathfrak{S}_N} \Phi(P, QP^{-1}) e^{i<P\mathbf{k},Q\mathbf{x}>}, \quad x_{q(1)} < \dots < x_{q(N)} \quad (6)$$

Periodic boundary conditions in the initial problem yield the first auxiliary problem:

$$S_{j+1j}\dots S_{Nj} S_{1j}\dots S_{j-1j}\phi = \Lambda_j \phi$$

Using again the *coordinate Bethe ansatz* and the periodicity conditions, we find the second auxiliary problem with the Hamiltonian being the chain of permutations. The obtained *Bethe equations* resemble the Lieb-Wu ones with an additional phase. We write them in the next panel.

$gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$ model

Generalizing the previous model we take an example with $\mathfrak{n}+\mathfrak{m}+1$ different types of "particle": $2\uparrow, \dots, (\mathfrak{n}+\mathfrak{m})\uparrow$ and $2\downarrow$ on a vacuum state with the choice of *projectors*: $C_{\uparrow k} = E^{11}_{\uparrow k} - \sum_{a=2}^{\mathfrak{n}+\mathfrak{m}} E^{aa}_{\uparrow k}$ and $C_{\downarrow k} = E^{11}_{\downarrow k} - E^{22}_{\downarrow k}$

The *Bethe equations* are

$$e^{ik_jL} = (-1)^{K+N+1}\prod_{m=1}^{K}\frac{i\sin k_j + ia_m + \frac{u}{4}}{i\sin k_j + ia_m - \frac{u}{4}}, \quad j\in[1,N] \quad (7)$$

$$(-1)^N \prod_{j=1}^{N}\frac{i\sin k_j + ia_m + \frac{u}{4}}{i\sin k_j + ia_m - \frac{u}{4}} = \Lambda(\vec{n}^{(3)}) \prod_{l=1,\, l\neq m}^{K}\frac{ia_m - ia_l + \frac{u}{2}}{ia_m - ia_l - \frac{u}{2}}, \quad m\in[1,K] \quad (8)$$

$$\Lambda(\vec{n}^{(3)}) = \exp\left(\frac{2i\pi}{K}\sum_{i=1}^{M} n^{(3)}_i\right), \quad 1 \le n^{(3)}_1 < \dots < n^{(3)}_M \le K \;\text{ and }\; M\in[0,K] \quad (9)$$

where

- L is the number of sites considered in the Hubbard model
- N is total number of $2\downarrow, 2\uparrow, 3\uparrow, \dots, (\mathfrak{n}+\mathfrak{m})\uparrow$ "particles".
- K counts the total number of excitations from $2\uparrow$ to $(\mathfrak{n}+\mathfrak{m})\uparrow$
- M numbers the $3\uparrow, \dots, (\mathfrak{n}+\mathfrak{m})\uparrow$ "particles".

The Bethe parameters $n^{(k)}_i$, for each particle $k\uparrow$, $3 < k \le \mathfrak{m}+\mathfrak{n}$, don't show up in the Bethe equations.

$gl(2|2) \oplus gl(2)$ model

Now we take an example with a different choice of *projectors* from the previous examples: $C_{\uparrow k} = E^{11}_{\uparrow k} + E^{22}_{\uparrow k} - E^{33}_{\uparrow k} - E^{44}_{\uparrow k}$ and $C_{\downarrow k} = E^{11}_{\downarrow k} - E^{22}_{\downarrow k}$ and we have 4 types of particle: $2\uparrow, 2\downarrow, 3\uparrow$ and $4\uparrow$ on a vacuum state.

This choice of projectors means that together with Hubbard "electron"-like particles we have another sort of "heavy" particle in interaction. Using again the *coordinate Bethe Ansatz* approach, we first introduce $\mathbb{A} = \{a_1, a_2, \dots, a_{N_1}\}$ for some integers such that $1 \le a_1 < a_2 < \dots < a_{N_1} \le N$. Then, the *Bethe equations* can be written as

$$e^{ik_j(L-N_2-N_3)} = (-1)^{N_1-1} \;\text{ for }\; j\in\mathbb{A}, \quad\text{and}\quad e^{ik_jL} = (-1)^{N+1-(N_1+N_2+N_3)}\prod_{m=1}^{N_2+N_3}\frac{i\sin k_j + ib_m + \frac{u}{4}}{i\sin k_j + ib_m - \frac{u}{4}}, \;\text{ for }\; j\in[1,N]\setminus\mathbb{A} \quad (10)$$

$$(-1)^{N-N_1}\prod_{\substack{j=1\\ j\notin\mathbb{A}}}^{N}\frac{i\sin k_j + ib_m + \frac{u}{4}}{i\sin k_j + ib_m - \frac{u}{4}} = \Lambda(\vec{n})\prod_{j\in\mathbb{A}} e^{-ik_j}\prod_{\substack{l=1\\ l\neq m}}^{N_2+N_3}\frac{ib_m - ib_l + \frac{u}{2}}{ib_m - ib_l - \frac{u}{2}}, \;\text{ with }\; \Lambda(\vec{n}) = \exp\left(\frac{2i\pi}{N_2+N_3}\sum_{i=1}^{N_3} n_i\right) \quad (11)$$

$$\text{for}\quad m = 1,\dots,N_2+N_3, \;\text{ and }\; 1 \le n_1 < n_2 < \dots < n_{N_3} \le N_2+N_3$$

where

- L is the number of sites considered in the Hubbard model
- N is total number of all $2\uparrow, 2\downarrow, 3\uparrow$ and $4\uparrow$ "particles"
- N_1 counts $2\uparrow$ excitations, N_2, N_3 count respectively $3\uparrow$ and $4\uparrow$ particles.

We remark that with respect to the Bethe equations computed in the previous sections, the phase $\Lambda(\vec{n})$ has been changed to $\Lambda(\vec{n})\prod_{j\in\mathbb{A}} e^{-ik_j}$ showing a (partial) dependence on the momenta of the particles.