Beyond cusp anomalous dimension from integrability

Abstract: We study the sub-leading corrections to the cusp anomalous dimension in the high spin expansion of finite twist operators in N=4 SYM theory. Since they are still governed by a linear integral equation, we are able to carefully study the weak and strong coupling regimes by means of analytic and numeric methods. We pay particular attention to the strong coupling regime, where we observe the emergence of the mass of the O(6) NLSM in the sub-leading logarithmic terms.

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Calculation of the anomalous dimension of single trace operators in the sl(2) sector beyond l.o.

Spin, Twist

$$\mathrm{Tr}(\mathcal{D}^s\,\mathcal{Z}^L)+\dots \qquad \Delta - s - L = \gamma(g) = f(g)\log s + \dots$$

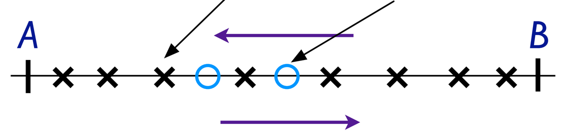
Strategy

- Step 1: Alternative formulation of the NLIE for the counting function
- Step 2: Eq. for the density of Bethe roots from the asymptotic BA in the large spin limit
- Step 3: Systematic expansion in powers of *ln s* and Neumann expansion
- Step 4: Study of the resulting infinite linear system with numeric and analytic methods

1 — NLIE for the Counting Function

$$Z(u) = \Phi(u) - \sum_{k=1}^{s}\phi(u,u_k)$$

- The Bethe roots are located in a finite interval [A,B]
- There is a finite number of holes
- Generic rapidity-dependence kernel



NLIE for the counting function

$$Z(u) = F(u) + 2(G\star L)(u)\,, \qquad L(u) = \mathrm{Im}\ln\left[1+e^{iZ(u+i0)}\right], \quad \varphi(u,v)=\frac{1}{2\pi}\frac{d}{dv}\phi(u,v)$$

$$(\varphi\star f)(u) = \int_A^B dv\,\varphi(u,v)f(v)\,,$$

Linear Integral Equations for Forcing term & Kernel

$$F(u) = f(u) - \int_A^B dv\varphi(u,v)F(v) \qquad G(u,v) = \varphi(u,v) - \int_A^B dw\varphi(u,w)G(w,v)$$

Observable eigenvalues

$$\sum_{k=1}^{s} O(u_k) = -\frac{1}{2\pi}\left[O(B)Z(B)-O(A)Z(A)\right] + \frac{1}{\pi}\left\{O(B)\mathrm{Im}\ln\left[1+e^{iZ(B)}\right] - O(A)\mathrm{Im}\ln\left[1+e^{iZ(A)}\right]\right\} +$$
$$+\int_A^B \frac{dv}{2\pi}O'(v)F(v) - \sum_{h=1}^{H_i}O(u_h^{(i)}) + 2\int_A^B\frac{dv}{2\pi}O'(v)\int_A^B dw\left[G(v,w)-\delta(v-w)\right]\mathrm{Im}\ln\left[1+e^{iZ(w-i0)}\right]$$

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Asymptotic BA for sl(2) sector

$$\left(\frac{u_k+i/2}{u_k-i/2}\right)^L \left(\frac{1+\frac{g^2}{2x_k^{-2}}}{1+\frac{g^2}{2x_k^{+2}}}\right)^L = \prod_{j\neq k=1}^{M}\frac{u_k-u_j-i}{u_k-u_j+i}\left(\frac{1-\frac{g^2}{2x_k^+x_j^-}}{1-\frac{g^2}{2x_k^-x_j^+}}\right)^2 e^{2i\theta(u_k,u_j)}$$

Dressing factor

$$x_k^{\pm} = x^{\pm}(u_k) = x(u_k\pm i/2) \qquad x(u) = \frac{u}{2}\left[1+\sqrt{1-\frac{2g^2}{u^2}}\right]$$

Linear Integral Equations for the forcing term

Splitting in one- and higher-loop terms $\quad F(u) = F_0(u) + F^H(u)$

$$F_0(u) = -2L\arctan 2u - (s+L-2)\left[\arctan(u-b_0)+\arctan(u+b_0)\right] + 2\sum_{h=1}^{L-2}\arctan(u-u_h^{(i)}) + \int_{-b_0}^{b_0}dv\frac{1}{\pi}\frac{1}{1+(u-v)^2}F_0(v)$$

$$F^H(u) = -iL\ln\left(\frac{1+\frac{g^2}{2x^-(u)^2}}{1+\frac{g^2}{2x^+(u)^2}}\right) - 2i\sum_{h=1}^{L-2}\left[\ln\left(\frac{1-\frac{g^2}{2x^+(u)x^-(u_h^{(i)})}}{1-\frac{g^2}{2x^-(u)x^+(u_h^{(i)})}}\right) + i\theta(u,u_h^{(i)})\right] - \frac{i}{\pi}Z(b)\left[\ln\left(\frac{1-\frac{g^2}{2x^+(u)x^-(b)}}{1-\frac{g^2}{2x^-(u)x^+(b)}}\right) + \ln\left(\frac{1-\frac{g^2}{2x^+(u)x^-(-b)}}{1-\frac{g^2}{2x^-(u)x^+(-b)}}\right) + i\theta(u,b) + i\theta(u,-b)\right] +$$
$$+\int_{-b}^{b}\frac{dv}{\pi}\frac{1}{1+(u-v)^2}F^H(v) + \frac{i}{\pi}\int_{-b}^{b}dv\left[\frac{d}{dv}\ln\left(\frac{1-\frac{g^2}{2x^+(u)x^-(v)}}{1-\frac{g^2}{2x^-(u)x^+(v)}}\right) + i\frac{d}{dv}\theta(u,v)\right]\left[F_0(v)+F^H(v)\right]$$

Anomalous dimension

$$E(g,s,L) = \int_{-b_0}^{b_0}\frac{dv}{2\pi}\frac{d}{dv}\left[\frac{i}{x^+(v)}-\frac{i}{x^-(v)}\right]F_0(v) - (L-2)\left[\frac{i}{x^+(0)}-\frac{i}{x^-(0)}\right] + \int_{-\infty}^{+\infty}\frac{dv}{2\pi}\frac{d}{dv}\left[\frac{i}{x^+(v)}-\frac{i}{x^-(v)}\right]F^H(v) + o(s^0)$$

- In the large spin limit the anomalous dimension depends on the forcing term *F(u)* only
- The anomalous dimension is calculated by solving a couple of LIE
- The simplification is due to the large spin limit

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- Rewrite everything in terms of the density of Bethe roots d/du F(u)
- Passing to Fourier space and redefinition of the density
- These equations are exact up to order O(1/s)

$$\gamma(g,s,L) = \frac{1}{\pi}\lim_{k\to 0}\hat\sigma_H(k) + \dots$$

$$S(k) = \frac{L}{|k|}\left[1-J_0(\sqrt{2}gk)\right] + \frac{1}{\pi|k|}\int_{-\infty}^{+\infty}\frac{dh}{|h|}\Big[\sum_{r=1}^{\infty}r(-1)^{r+1}J_r(\sqrt{2}gk)J_r(\sqrt{2}gh)\frac{1-\mathrm{sgn}(kh)}{2}e^{-\frac{|h|}{2}} + \qquad S(k) = \frac{2\sinh\frac{|k|}{2}}{2\pi|k|}\hat\sigma_H(k)$$
$$+\mathrm{sgn}(h)\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}c_{r,r+1+2\nu}(g)(-1)^{r+\nu}e^{-\frac{|h|}{2}}\left(J_{r-1}(\sqrt{2}gk)J_{r+2\nu}(\sqrt{2}gh) - J_{r-1}(\sqrt{2}gh)J_{r+2\nu}(\sqrt{2}gk)\right)\Big]\cdot$$
$$\cdot\left\{\frac{\pi|h|}{\sinh\frac{|h|}{2}}S(h) - 4\pi\ln 2\,\delta(h) - \pi(L-2)\frac{1-e^{\frac{|h|}{2}}}{\sinh\frac{|h|}{2}} - 2\pi\frac{1-e^{-\frac{|h|}{2}}\cos\frac{hs}{\sqrt{2}}}{\sinh\frac{|h|}{2}} + \pi\frac{e^{-\frac{|h|}{2}}}{\sinh\frac{|h|}{2}}\sum_{l=1}^{L-2}\left[e^{-ihu_l}-1\right] + 2\pi i\frac{e^{-\frac{|h|}{2}}}{\sinh\frac{|h|}{2}}\sum_{l=1}^{L-2}\sin\left(hu_l^{(0)}\right)\right\}$$

holes at large spin

- Neumann expansion and formulation in terms of an infinite linear system satisfied by the Neumann modes $(u_h, u_h^{(0)} = 0)$

$$S_r(g) = S_r^{BES}(g)\ln s + (L-2)\,S_r^{(1)}(g) + S_r^{extra}(g)$$

$$S(k) = \sum_{p=1}^{\infty}S_p(g)\frac{J_p(\sqrt{2}gk)}{k}$$

$$S_{2p-1}^{extra}(g) = 2\sqrt{2}g\gamma_E\delta_{p,1} + 4(2p-1)\int_0^{\infty}\frac{dh}{h}\frac{\tilde J_{2p-1}(\sqrt{2}gh)}{e^h-1} - 2(2p-1)\sum_{m=1}^{\infty}Z_{2p-1,m}(g)S_m^{extra}(g)$$

$$S_{2p}^{extra}(g) = 4 + 8p\int_0^{\infty}\frac{dh}{h}\frac{J_{2p}(\sqrt{2}gh)}{e^h-1} + 4p\sum_{m=1}^{\infty}Z_{2p,2m-1}(g)S_{2m-1}^{extra}(g) - 4p\sum_{m=1}^{\infty}Z_{2p,2m}(g)S_{2m}^{extra}(g)$$

$$\gamma(g,s,L) = 2S(0) = \sqrt{2}g\,S_1(g)$$

Universal scaling function [2] — First generalised s.f. [2, 3]

$$\gamma(g,s,L) = f(g)\ln s + (L-2)f_1(g) + f_{sl}(g)$$

$u_h, u_h^{(0)} = 0$

When the holes term is switched on, in the large spin limit we have:

Condition to be imposed at one- and all-loop

$$F(u_h) = \pi(2h+1-L)\,, \qquad h = 1,\dots,L-2$$

Holes position at large spin

$$u_h^{(0)} = -\frac{\pi(2h+1-L)}{4\ln s}\left[1 - \frac{L\ln 2+\gamma_E}{\ln s} + \left(\frac{L\ln 2+\gamma_E}{\ln s}\right)^2\right] + O\left(\frac{1}{\ln^4 s}\right)$$

$$u_h = \frac{\pi(2h+1-L)}{\sigma_1^{(0)}\ln s}\left[1 - \frac{\sigma_0^{(0)}}{\sigma_1^{(0)}\ln s} + \frac{1}{\ln^2 s}\left(\frac{\sigma_{-1}^{(0)}}{\sigma_1^{(0)}} - \left(\frac{\sigma_0^{(0)}}{\sigma_1^{(0)}}\right)^2 - \pi^2(2h+1-L)^2\frac{\sigma_1^{(2)}}{(\sigma_1^{(0)})^3}\right)\right] + O\left(\frac{1}{\ln^4 s}\right)$$

$\sigma_k^{(m)}(g) \longrightarrow$ *m-th* derivative of the density of Bethe roots at *u=0*, at order *k* in the *ln s* expansion

$$\gamma(g,s,L) = f(g)\ln s + (L-2)f_1(g) + f_{sl}(g) + \sum_{n=1}^{\infty}\gamma^{(n)}(g,L)\,\ln^{-n}s + O(1/s)$$

- The subleading log terms are given by linear systems of equations similar to those found in [6]
- The leading strong coupling behaviour can be evaluated analytically

$$\gamma^{(1)}(g,L) \equiv 0 \quad \leftarrow \text{Identically zero } \forall g$$

$$\gamma^{(2)}(g,L) = \frac{\pi^2}{24\,m(g)}(L-3)(L-2)(L-1) + \dots$$

$$\gamma^{(3)}(g,L) = -\frac{\pi}{3\,m^2(g)}\ln g\,(L-3)(L-2)(L-1) + \dots$$

$$\gamma^{(4)}(g,L) = -\frac{\pi^4(L-3)(L-2)(L-1)}{16\,m^3(g)}\left\{\frac{2\ln^2 g}{9\pi^2} + \dots\right\} + \dots$$

NEW!

The mass *m(g)* of the O(6) NLSM appears explicitly

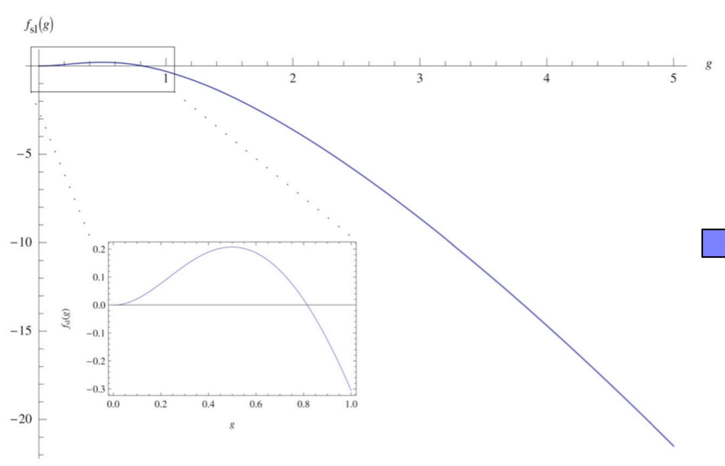
4 — Numerical solution of the (truncated [5]) linear system [4]

- Best fit at strong coupling

$$f_{sl}(g) = f(g)\ln\frac{2\sqrt{2}}{g} + k_1 g + k_0 + \frac{k_{-1}}{g} + O(1/g^2)$$

$$k_1 = -2.828426 \pm 0.000001$$
$$k_0 = 0.3238 \pm 0.0001$$
$$k_{-1} = -0.01194 \pm 0.00015$$

Full agreement with [7]



- Strong coupling expansion

$$f_{sl}(g) = 2\sqrt{2}\,g\left[\ln\frac{2\sqrt{2}}{g} - \frac{k_1}{2\sqrt{2}} - \frac{3\ln 2}{2\sqrt{2}\pi\,g}\ln\frac{2\sqrt{2}}{g} + \frac{k_0}{2\sqrt{2}\,g} - \frac{\mathrm{K}}{8\pi^2g^2}\ln\frac{2\sqrt{2}}{g} + \frac{k_{-1}}{2\sqrt{2}\,g^2} + O\left(\frac{\ln g}{g^3}\right)\right]$$

Weak coupling expansion [4]

$$f_{sl}(g) = \gamma_E f(g) - 24\zeta(3)\left(\frac{g}{\sqrt{2}}\right)^4 + \frac{16}{3}\left(\pi^2\zeta(3)+30\zeta(5)\right)\left(\frac{g}{\sqrt{2}}\right)^6 +$$
$$-\frac{8}{15}\left(7\pi^4\zeta(3)+50\pi^2\zeta(5)+2625\zeta(7)\right)\left(\frac{g}{\sqrt{2}}\right)^8 +$$
$$+\left(\frac{128}{35}\pi^6\zeta(3)+192\zeta(3)^3+\frac{832}{45}\pi^4\zeta(5)+\frac{560}{3}\pi^2\zeta(7)+14128\zeta(9)\right)\left(\frac{g}{\sqrt{2}}\right)^{10} +$$
$$-\frac{8}{14175}\Big(7319\pi^8\zeta(3)+33300\pi^6\zeta(5)+229320\pi^4\zeta(7)+47250\pi^2\left(4\zeta(3)^3+59\zeta(9)\right)+$$
$$+113400\left(82\zeta(3)^2\zeta(5)+2439\zeta(11)\right)\Big)\left(\frac{g}{\sqrt{2}}\right)^{12} + \dots$$