Classification of Yangians of Lie Superalgebras and their R-matrices

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Motivation

- Yangians of superalgebras appearing in many physical systems, i.e. AdS/CFT
- Derive an abstract, representation-independent form of R-matrix for Yangians of Lie superalgebras
- Derive fundamental R-matrix for simple Lie superalgebras including complicated dressing factors

Notations

- Lie algebras and its subalgebras are denoted by small gothic characters $\mathfrak{e}, \mathfrak{h}, \ldots$
- Lie algebra and Yangian generators are denoted by capital gothic characters $\mathfrak{E}, \mathfrak{H}, \mathfrak{K}, \ldots$
- $E_{i,j}$ denotes an $n \times n$ matrix with the only nonzero entry at (i, j) position.
- Although we only consider classical Lie algebras here, we will need the notion of q-numbers

$$n \to [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$$

- The bracket $(.)_m$ denotes m-th coefficient in the Taylor expansion

$$\Big(f(x)\Big)_{m+1} = \left(\frac{d^m f(x)}{dx^m}\right)_{x=0}$$

Yangian in RTT Realization

- Consider Lie (super)algebra $\mathfrak{gl}(n|m)$ and its vector representation
- Yangian $Y(\mathfrak{gl}(n|m))$ is isomorphic to associative algebra $U(R)$ generated by $\mathbb{1}$ and the matrices

$$T_{ij}^{(k)}, \quad i, j = \overline{1, n+m}, \quad k \in \mathbb{Z}_{\geq 0}$$

It is convenient to gather them in the formal series

$$T(\lambda) = \sum_{i,j=1}^{n} \sum_{n=0}^{+\infty} T_{ij}^{(n)} \lambda^{-n} E_{i,j}\,,$$

$T(\lambda)$ satisfy the so-called RTT relations

$$R^{(n)}(\lambda - \mu)(T(\lambda) \otimes 1)(1 \otimes T(\mu)) = (1 \otimes T(\mu))(T(\lambda) \otimes 1)R^{(n)}(\lambda - \mu)\,,$$
$$\mathrm{qdet}(T(\lambda)) = 1\,,$$

where qdet is the quantum determinant and the Yang matrix is given by

$$R^{(n)}(\lambda) = 1 \otimes 1 + \sum_{1 \leq i,j \leq n} \lambda^{-1} E_{i,j} \otimes E_{j,i}$$

- Commutation relations for $T(\lambda)$

$$(\lambda - \mu)[T_{ij}(\lambda), T_{kl}(\mu)] = T_{kj}(\mu)T_{il}(\lambda) - T_{kj}(\lambda)T_{il}(\mu)$$

- $T_{ij}(\lambda)$ is a generating function for the Yangian $Y(\mathfrak{gl}(n|m))$ generators. Expansion around $\lambda = \infty$ gives these generators and commutation relations on $T_{ij}(\lambda)$ give defining relations on Yangian generators as well as Serre relations. Coproduct for Yangian generators follow from coproduct of $T_{ij}(\lambda)$.
- Call the diagonal and upper/lower triangular part of $T_{ij}^{(k)}$ $\mathfrak{H}_i^{(k)}, \mathfrak{E}_i^{(k)}, \mathfrak{F}_i^{(k)} | i = \overline{1, n+m-1}, k \in \mathbb{Z}_{>}$, then from RTT defining relations it follows

$$[\mathfrak{H}_i^{(0)}, \mathfrak{E}_j^{(l)}] = A_{ij}\mathfrak{E}_j^{(l)}, \quad [\mathfrak{H}_i^{(0)}, \mathfrak{F}_j^{()l}] = -A_{ij}\mathfrak{F}_j^{(l)} \ldots$$

- These can be also taken as the abstract defining relations of the Yangian
- Representation of the Chevalley-Serre basis:

$$\begin{aligned}
&1 \leq i < n: \quad \mathfrak{H}_i = E_{i,i} - E_{i+1,i+1}, \mathfrak{E}^+ = E_{i,i+1}, \mathfrak{E}^- = E_{i+1,i}\\
&i = n: \quad \mathfrak{H}_n = E_{n,n} + E_{n+1,n+1}, \mathfrak{E}^+ = E_{n,n+1}, \mathfrak{E}^- = E_{n+1,n}\\
&n < i < n+m: \quad \mathfrak{H}_i = -E_{i,i} + E_{i+1,i+1}, \mathfrak{E}^+ = E_{i,i+1}, \mathfrak{E}^- = -E_{i+1,i}
\end{aligned}$$

- For the case $n = m$, the additional Cartan generator $\mathfrak{H}_{2n}$ must be introduced

$$\mathfrak{H}_{2n} = \frac{1}{2}\left(\sum_{i=1}^{n} E_{i,i} - \sum_{i=n+1}^{2n} E_{i,i}\right)$$

whereas for $n \neq m$ one may drop the unessential identity matrix.
- The non-simple positive/negative roots: remaining matrices

$$E_{i,j}, i < j - 1, \qquad E_{i,j}, i - 1 > j$$

Quantum Double

- Algebraically, R-matrix is the canonical element of the Hopf Algebra tensored with its dual (similar to a Casimir)
- Classical analogy: Lie algebra $\mathfrak{g}$ with generators $[\mathfrak{J}^a, \mathfrak{J}^b] = f_c^{ab}\mathfrak{J}^c$ extends to loop algebra (Kac-Moody algebra without central charge) $\mathfrak{g}[\lambda, \lambda^{-1}]$ with generators $[\mathfrak{J}_n^a, \mathfrak{J}_m^b] = f_c^{ab}\mathfrak{J}_{n+m}^c$, i.e. $\mathfrak{J}_n^a = \lambda^n \mathfrak{J}^a$. Then Killing form $\kappa^{ab} \propto str(\mathfrak{J}^a, \mathfrak{J}^b)$ is extended by $(\mathfrak{J}_n^a, \mathfrak{J}_m^b) = \kappa^{ab}\delta_{n,-m-1}$. This form splits $\mathfrak{g}[\lambda, \lambda^{-1}] = \mathfrak{g}[\lambda] + \lambda^{-1}\mathfrak{g}[\lambda^{-1}]$ into positive and negative degrees.
- Classical r-matrix:

$$r = \sum_{n=0}^{\infty} \kappa_{ab}\mathfrak{J}_n^a \otimes \mathfrak{J}_{-n-1}^b$$

- Quantum R-matrix of Yangian: $\mathcal{R} = \sum_{J \in \mathcal{Y}(\mathfrak{g})} J \otimes J^*$, where J^* is the dual of J
- Invariant form for Yangian:

$$\begin{aligned}
\left(\mathfrak{E}_{i,k}^+, \mathfrak{E}_{j,l}^-\right) &= -\delta_{ij}\delta_{k,-l-1}\\
\left(\mathfrak{E}_{i,k}^-, \mathfrak{E}_{j,l}^+\right) &= -(-1)^{|i|}\delta_{ij}\delta_{k,-l-1}\\
(\mathfrak{H}_{i,k}, \mathfrak{H}_{j,-l-1}) &= -2\left(\frac{A_{ij}}{2}\right)^{n-m}\binom{n}{m}, \quad n \geq m,
\end{aligned}$$

or, in terms of generating function for the Cartan part,

$$\left(\mathfrak{H}_i^+(\lambda), \mathfrak{H}_j^-(\tilde{\lambda})\right) = \frac{\lambda - \tilde{\lambda} + \frac{A_{ij}}{2}}{\lambda - \tilde{\lambda} - \frac{A_{ij}}{2}}$$

- For explicit form of R-matrix one needs to diagonalise this form.

R-matrix

- For a simple Lie superalgebra $\mathfrak{g}$ with symmetrized Cartan matrix $A^{\mathfrak{g}}$ we define its quantum counterpart by

$$A_{ij}^{\mathfrak{g}} \to A_{ij}^{\mathfrak{g}}(q) := \left[A_{ij}^{\mathfrak{g}}\right]_q$$

- The following matrix is of great importance, while constructing the universal R-matrix of $\mathfrak{g}$

$$C_{ij}^{\mathfrak{g}}(q) = \ell^{\mathfrak{g}}(q)\left(A^{\mathfrak{g}}(q)\right)_{ij}^{-1}.$$

- The constant $\ell^{\mathfrak{g}}(q)$ is defined as the *minimal* proportionality factor that makes $C^{\mathfrak{g}}(q)$ polynomial in q and q^{-1}. It is usually proportional to the dual coxeter number. However, for $\mathfrak{gl}(n|n)$ the dual coxeter number is zero and we have $\ell^{\mathfrak{g}}(0) = n$. In what follows $\ell^{\mathfrak{g}}(0) \equiv \ell^{\mathfrak{g}}$.
- Triangular decomposition of $\mathfrak{g}$ into subalgebras of positive roots, Cartan and negative roots

$$\mathfrak{g} = \mathfrak{e}^+ \oplus \mathfrak{h} \oplus \mathfrak{e}^-,$$

one has $[\mathfrak{e}_\pm, \mathfrak{h}] \subset \mathfrak{e}_\pm$. We denote Δ^+ be the space of positive roots of $\mathfrak{g}$.
- Triangular decomposition of Lie algebra induces similar decomposition of double Yangian $\mathcal{DY}(\mathfrak{g})$ which entails triangular decomposition of universal R-matrix

$$\mathcal{R}_{12} = \mathcal{R}_+\mathcal{R}_H\mathcal{R}_-.$$

The $\mathcal{R}_+$ and $\mathcal{R}_-$ are defined through

$$\mathcal{R}_+ = \prod_{\alpha \in \Xi^+}^{\rightarrow} \exp(-(-1)^{\theta(\alpha)}a(\alpha)\mathfrak{E}_\alpha^+ \otimes \mathfrak{E}_\alpha^-),$$
$$\mathcal{R}_- = \prod_{\alpha \in \Xi^+}^{\leftarrow} \exp(-(-1)^{\theta(\alpha)}a(\alpha)\mathfrak{E}_\alpha^- \otimes \mathfrak{E}_\alpha^+),$$

with $\theta(\alpha)$ being the parity of $\mathfrak{E}_\alpha^\pm$. The set of positive roots is defined by

$$\Xi^+ := \{\gamma + n\delta | \gamma \in \Delta^+\},$$

where δ denotes the affine root and

$$[\mathfrak{E}_\alpha^+, \mathfrak{E}_\alpha^-] = a(\alpha)^{-1}\mathfrak{H}_\gamma, \quad \alpha = \gamma + n\delta, \quad \gamma \in \Delta_+(\mathfrak{g})$$

The part corresponding to the Cartan part of the Yangain $\mathcal{R}_H$ reads

$$\prod_{n=0}^{\infty} \exp\left(\left(\mathfrak{K}_{i,+}'(\lambda)\right)_m \otimes \left(C_{i,j}^{\mathfrak{g}}(T^{1/2})\mathfrak{K}_{j,-}(\tilde{\lambda} + \ell^{\mathfrak{g}}(n+1))\right)_{m+1}\right)$$

The dependence on $\mathfrak{g}$ has been highlighted in red. The symbols $\mathfrak{K}_i^\pm$ are shorthand notations for the Drinfeld polynomials

$$\mathfrak{K}_i^\pm(\lambda) = \log \mathfrak{H}_i^\pm(\lambda),$$

where

$$\mathfrak{H}_i^+(\lambda) = 1 + \sum_{n=0}^{\infty} \mathfrak{H}_{i,n}\lambda^{-n-1}, \quad \mathfrak{H}_i^-(\lambda) = 1 - \sum_{n=-1}^{-\infty} \mathfrak{H}_{i,n}\lambda^{-n-1}.$$

The case of $\mathfrak{gl}(n|m)$, $\quad n \neq m$

- The q-Cartan matrix for the distinguished Dynkin diagram is given by

$$A^{\mathfrak{gl}(n|m)}(q) = \begin{pmatrix}
[2]_q & -1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\
-1 & [2]_q & -1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots\\
0 & \ddots & \ddots & -1 & 0 & \ldots & \ldots & \ldots & \ldots\\
\vdots & \ldots & -1 & [2]_q & -1 & \ddots & \ldots & \ldots & \ldots\\
0 & \ldots & 0 & -1 & 0 & 1 & 0 & \ldots & \ldots\\
\vdots & \ldots & \ldots & \ldots & 1 & -[2]_q & 1 & 0 & \ldots\\
\vdots & \ldots & \ldots & \ldots & 0 & 1 & -[2]_q & 1 & \ddots\\
\vdots & \ldots & \ldots & \ldots & \vdots & \ldots & \ddots & \ddots & 1\\
\vdots & \ldots & \ldots & \ldots & 0 & \ldots & \ldots & 1 & -[2]_q
\end{pmatrix}$$

- Its inverse $\left(A^{\mathfrak{gl}(n|m)}(q)\right)^{-1}$ is given by

$$\begin{pmatrix}
a_{n+m-1,1} & \ldots & \ldots & \ldots & \text{upper} & \text{elements} & \text{are}\\
\vdots & \ddots & \ldots & \ldots & \ldots & \text{obtained} & \text{by}\\
a_{m+1,1} & \ldots & a_{m+1,n-1} & \ldots & \ldots & \ldots & i \leftrightarrow j\\
b_{m,1} & \ldots & \ldots & b_{m,n} & \ldots & \ldots & \ldots\\
\vdots & \ddots & \ldots & \vdots & c_{m-1,n+1} & \ldots & \ldots\\
b_{2,1} & b_{2,2} & \ldots & \vdots & \vdots & \ddots & \ldots\\
b_{1,1} & b_{1,2} & \ldots & b_{1,n} & c_{1,n+1} & \ldots & c_{1,n+m-1}
\end{pmatrix}$$

with

$$\begin{aligned}
a_{i,j} &= -\frac{[2m - i]_q[j]_q}{[n - m]_q},\\
b_{i,j} &= -\frac{[i]_q[j]_q}{[n - m]_q},\\
c_{i,j} &= -\frac{[i]_q[2n - j]_q}{[n - m]_q}
\end{aligned}$$

The case of $\mathfrak{gl}(n|n)$

- In the case $n = m$ the Cartan matrix is singular and need to be extended to

$$A^{\mathfrak{gl}(n|n)}(q) = \begin{pmatrix}
[2]_q & -1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0\\
-1 & [2]_q & -1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \vdots\\
0 & \ddots & \ddots & -1 & 0 & \ldots & \ldots & \ldots & \ldots & \vdots\\
\vdots & \ldots & -1 & [2]_q & -1 & \ddots & \ldots & \ldots & \ldots & 0\\
0 & \ldots & 0 & -1 & 0 & 1 & 0 & \ldots & \ldots & 1\\
\vdots & \ldots & \ldots & \ldots & 1 & -[2]_q & 1 & 0 & \ldots & 0\\
\vdots & \ldots & \ldots & \ldots & 0 & 1 & -[2]_q & 1 & \ddots & \vdots\\
\vdots & \ldots & \ldots & \ldots & \vdots & \ldots & \ddots & \ddots & 1 & \vdots\\
\vdots & \ldots & \ldots & \ldots & 0 & \ldots & \ldots & 1 & -[2]_q & 0\\
0 & \ldots & \ldots & 0 & 1 & 0 & \ldots & \ldots & 0 & \lambda
\end{pmatrix}$$

The case of $\mathfrak{gl}(n|n)$ continued

- The corresponding inverse matrix $\left(A^{\mathfrak{gl}(n|n)}(q)\right)^{-1}$ has a similar structure to the $\mathfrak{gl}(n|m)$ case, with the exception that the the last row and column are distinguished

$$\begin{pmatrix}
a_{2n-1,1} & \ldots & \ldots & \ldots & \ldots & \text{upper} & \text{elements} & \text{are}\\
\vdots & \ddots & \ldots & \ldots & \ldots & \ldots & \text{obtained} & \text{by}\\
a_{n+1,1} & \ldots & a_{n+1,n-1} & \ldots & \ldots & \ldots & \ldots & i \leftrightarrow j\\
b_{n,1} & \ldots & \ldots & b_{n,n} & \ldots & \ldots & \ldots & \vdots\\
\vdots & \ddots & \ldots & \vdots & c_{n-1,n+1} & \ldots & \ldots & \vdots\\
b_{2,1} & b_{2,2} & \ldots & \vdots & \vdots & \ddots & \ldots & \vdots\\
b_{1,1} & b_{1,2} & \ldots & b_{1,n} & c_{1,n+1} & \ldots & c_{1,2n-1} & \vdots\\
d_{0,1} & d_{0,2} & \ldots & d_{0,n} & d_{0,n+1} & \ldots & \ldots & d_{0,2n}
\end{pmatrix}$$

The four parts of the matrix are defined as follows:

$$\begin{aligned}
a_{i,j} &= -\frac{[j]_q}{[n]_q^2}\left([i - n]_q[n]_q - \lambda[2n - i]_q\right),\\
b_{i,j} &= -\frac{[i]_q[j]_q}{[n]_q^2},\\
c_{i,j} &= -\frac{[i]_q}{[n]_q^2},\\
d_{0,j} &= \begin{cases} \frac{[j]_q}{[n]_q}, & 1 \leq j \leq n\\ \frac{[2n-j]_q}{[n]_q}, & n < j \leq 2n \end{cases}
\end{aligned}$$

Evaluating the R-matrix

- Fundamental representation of Yangian:

$$\mathfrak{H}_{i,k} = u_i^k \mathfrak{H}_{i,0}, \quad \mathfrak{E}_{i,k}^\pm = u_i^k \mathfrak{E}_{i,0}^\pm,$$

- The shifted spectral parameters are defined as follow

$$\begin{aligned}
1 \leq i \leq n: &\quad u_i = u + \frac{i}{2},\\
n \leq i < n+m: &\quad u_i = u + \frac{2n - i}{2}
\end{aligned}$$

- Evaluate the universal R-matrix on the fundamental representation
- The $\mathcal{R}_+$ and $\mathcal{R}_-$ parts read

$$\mathcal{R}_+ = \prod_{k=1,\ldots\frac{(n+m)(n+m-1)}{2}}^{\rightarrow} \exp\left(-\sum_{n=0}^{\infty} \mathfrak{E}_{\alpha_k,n}^+ \otimes \mathfrak{E}_{\alpha_k,-n-1}^-\right),$$
$$\mathcal{R}_- = \prod_{k=\frac{(n+m)(n+m-1)}{2},\ldots 1}^{\leftarrow} \exp\left(-\sum_{n=0}^{\infty} \mathfrak{E}_{\alpha_k,n}^- \otimes \mathfrak{E}_{\alpha_k,-n-1}^+\right)$$

- Due to the nilpotence of the factors, this simplifies to

$$\begin{aligned}
\exp\left(-\sum_{n=0}^{\infty} \mathfrak{E}_{\alpha_k,n}^+ \otimes \mathfrak{E}_{\alpha_k,-n-1}^-\right) &= \exp\left(-\sum_{n=0}^{\infty} u_i^n \mathfrak{E}_{\alpha_k,0}^+ \otimes \tilde{u}_i^{-n-1}\mathfrak{E}_{\alpha_k,0}^-\right),\\
&= \exp\left(\frac{1}{u - \tilde{u}}\mathfrak{E}_{\alpha_k,0}^+ \otimes \mathfrak{E}_{\alpha_k,0}^-\right) = 1 + \frac{1}{u - \tilde{u}}\mathfrak{E}_{\alpha_k,0}^+ \otimes \mathfrak{E}_{\alpha_k,0}^-
\end{aligned}$$

- The Cartan part consists of the generating functions

$$\begin{aligned}
\mathfrak{H}_{i,+}(\lambda) &= 1 + \frac{1}{\lambda - u_i}\mathfrak{H}_i,\\
\mathfrak{H}_{i,-}(\mu) &= 1 - \frac{1}{\tilde{u}_i - \tilde{\lambda}}\mathfrak{H}_i.
\end{aligned}$$

$$\begin{aligned}
\mathfrak{K}_{i,+}(\lambda)' = \log \mathfrak{H}_{i,+}(\lambda)' &= \frac{1}{\lambda - u_i - \mathfrak{H}_i} - \frac{1}{\lambda - u_i}\\
&= \sum_{n=0}^{\infty} \lambda^{-n-1}\left((u_i - \mathfrak{H}_i)^n - u_i^n\right),\\
\mathfrak{K}_{i,-}(\tilde{\lambda}) = \log \mathfrak{H}_{i,-}(\tilde{\lambda}) &= \log \frac{u_i - \mathfrak{H}_i}{u_i}\frac{\frac{\tilde{\lambda}}{u_i - \mathfrak{H}_i} - 1}{\frac{\tilde{\lambda}}{u_i} - 1} =\\
&\log \frac{u_i - \mathfrak{H}_i}{u_i} + \sum_{n=1}^{\infty} \frac{\left((\frac{\tilde{\lambda}}{u_i})^n - (\frac{\tilde{\lambda}}{u_i - \mathfrak{H}_i})^n\right)}{n}
\end{aligned}$$

- Plugging these expressions into $\mathcal{R}_H$ we get, for $\mathfrak{gl}(n|m), n \neq m$, Gamma functions. The final answer is given by

$$R = R_0\left(\frac{u - \tilde{u}}{u - \tilde{u} + 1} + \frac{1}{u - \tilde{u} + 1}\mathcal{P}\right),$$

where $\mathcal{P}$ is the graded $(n+m)^2 \times (n+m)^2$ dimensional permutation operator and

$$R_0 = \frac{\Gamma\left(\frac{u - \tilde{u} - 1}{m - n}\right)\Gamma\left(\frac{u - \tilde{u} + m - n + 1}{m - n}\right)}{\Gamma\left(\frac{u - \tilde{u}}{m - n}\right)\Gamma\left(\frac{u - \tilde{u} + m - n}{m - n}\right)}$$

- For $\mathfrak{gl}(n|n)$ we get $R_0 = \frac{u - \tilde{u} + \frac{1}{2}}{u - \tilde{u} - \frac{1}{2}}$

Conclusions and Outlook

- Explicit construction of universal R-matrix is elaborated. *En route* an explicit form of inverse Cartan matrix for $\mathfrak{gl}(n|m)$ algebra is obtained.
- R-matrix is explicitly computed in evaluation representation of $DY(\mathfrak{gl}(n|m))$. The computation is in agreement with known fundamental R-matrices.
- Construction works also in case of $\mathfrak{osp}(n|m)$, but representation theory is more difficult as there are no evaluation representations.

References

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