Wall-crossing in N=2 gauge theory and integrability

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DG, G.Moore, A.Neitzke: to appear

Introduction

Wall-crossing phenomena in N=2 field theory

- Why a talk in this workshop?
- Two reasons:
  - We explored SU(2) Hitchin system
    - Related to Sinh-Gordon
    - Applications to gluon scattering Alday, Maldacena
    - At least some methods should generalize
  - We met some TBA-like equations (or Y-system-like)
    - A coincidence? A hidden integrable system?
    - In any case, it should be fun to study or solve those equations
    - We also have fun differential equations. Could they be useful?

Outline

What is wall-crossing

Hyperkahler metrics and TBA-like equations

WKB analysis of SU(2) Hitchin system

N=2 4d gauge theories

N=2 4d gauge theories in the Coulomb branch

- Gauge multiplet has adjoint scalar
  - Expectation value Higgses to $U(1)^r$
  - r complex scalars u parametrize vacuum
  - r electric charges q_e, r magnetic charges q_m.
  - $<q,q'> = q_e q'_m - q_m q'_e$
- A BPS bound: M greater or equal to $|Z(q_e, q_m)|$
  - $Z= q_e a(u) + q_m a_D(u)$
    - periods (a,a_D) determine massless Lagrangian

Interesting massive spectrum of BPS particles

- $M = |Z(q_e, q_m)|$

BPS spectrum

BPS particles sit in reduced SUSY multiplets

- To become non-BPS, they must recombine
- Define a BPS degeneracy $\Omega(q,u)$
  - Naively, it should not vary with u

Exception: walls of marginal stability

- two particle states continuum: $M>M_1 + M_2$
- $q=q_1 + q_2$; $Z(q,u)=Z(q_1,u) + Z(q_2,u)$
- wall defined by $|Z(q,u)|=|Z(q_1,u)| + |Z(q_2,u)|$
  - BPS states can ``decay'' to continuum across walls

Wall-crossing

Are BPS spectra on the two sides of wall related?

- Near the wall, states which decay are very ``large'' in size
- Effective IR Lagrangian might know about decay

Wall crossing formula

- First attempts: Denef, Moore for two-particle decay
  - $\Delta\Omega(q_1 + q_2) = \langle q1,q2 \rangle\ \Omega(q_1)\ \Omega(q_2)$
- From related mathematical work, a full proposal
  - Kontsevich- Soibelmann wall crossing formula

KS wall crossing formula

Extremely surprising form

- Consider variables x_q, $x_{q+q'} = (-1)^{\langle q,q'\rangle} x_q \, x_{q'}$
- KS transformations K_q: $x_p \Rightarrow x_p(1-x_q)^{\langle q,p\rangle}$
- $\prod K_q^{\Omega(q)}$ in the order of arg $Z(q)$
- Overall product is unchanged across wall!
  - wall: arg $Z(q_1)$ = arg $Z(q_2)$
  - Order of arg $Z(q_1)$ and arg $Z(q_2)$ changes at wall
  - K_{q1} and K_{q2} do not commute
  - Change in $\Omega(q)$ follows

Example

Simplest wall:

- One electron, one monopole => electron,monopole,dyon
- $K_{(1,0)}K_{(0,1)} = K_{(0,1)}K_{(1,1)}K_{(1,0)}$
  - $K_{(0,1)}$: $[x_{(0,1)} , x_{(1,0)}] \Rightarrow [x_{(0,1)} , x_{(1,0)}(1-x_{(0,1)})]$
  - $K_{(1,0)}$: $[x_{(0,1)},x_{(1,0)}] \Rightarrow [x_{(0,1)}/(1-x_{(1,0)}) , x_{(1,0)}]$
  - $K_{(1,1)}$: $[x_{(0,1)},x_{(1,0)}] \Rightarrow [x_{(0,1)}/(1+x_{(0,1)} x_{(1,0)}) , x_{(1,0)}(1+x_{(0,1)} x_{(1,0)})]$
- A pentagon identity: $X_{n-1} X_{n+1}=1-X_n$ has period five

More interesting wall

- $K_{(2,-1)}K_{(0,1)} = K_{(0,1)} K_{(2,1)} K_{(4,1)} \ldots\ldots K_{(2,0)}^{-2} \ldots\ldots K_{(6,-1)} K_{(4,-1)} K_{(2,-1)}$
  - $X_{n-1} X_{n+1}=(1-X_n)^2$ has no period, but relation $\pm\infty$ is $K_{(2,0)}^{-2}$

A circle compactification

Pure U(1) theory on R^3 x S^1

- 4d: scalar a, gauge field A_i
  - moduli space: R^2
  - $a_D = \tau a$ τ is constant gauge coupling
- 3d: scalars a, $t = A_3$, t_D dual to 3d gauge field
  - moduli space: R^2 x T^2
  - It is (trivially) hyperkahler: S^2 worth of complex structures
    - complex coordinates in complex structure ζ?
      - » $X_e = \exp [R/\zeta\, a + i\, t + R\, \zeta\, a^*]$ $x_m = \exp [R/\zeta\, a_D + i\, t_D + R\, \zeta\, a_D^*]$
      - » R^2 x T^2 is C^* x C^*
    - Special $\zeta = 0$, $\zeta = \infty$
      - » a and $t_D - \tau t$ are holomorphic,
      - » R^2 x T^2 is C x elliptic curve

U(1) plus one massive particle

Loops of particle correct metric

- Correction strong when particle is light
- Correction shrinks t_D circle

4d loops: running coupling constant

- τ = log a/L
- Singular at a=0, codimension 2

3d loops: instantons from particle around S^1

- 3d masses: a, a*, t
- Codimension 3, and regular! ``periodic Taub-NUT''

Periodic Taub-NUT

Taub-NUT metric: hyperkahler circle fibration

- flat base space, fibration metric:
  - $ds^2 = V(x)\, dx^2 + V(x)^{-1}(dt_D + A)^2$
  - $V(x)$ is harmonic.

Periodic Taub-NUT

- $R^2 \times S^1$ base: a, a^*, t
- Single source at $a=t=0$.
  - Far away, V goes like $\log |a|/L$, $\tau = \log a/L$
- Regular at $a=t=0$.

Holomorphic functions

Complex coordinates in complex structure z?

- $\log x_e = R/\zeta\, a + i\, t + R\, \zeta\, a^*$
- $\log x_m = R/\zeta\, a_D + i\, t_D + R\, \zeta\, a_D^* +$

$$\left[\frac{iq}{4\pi}\int_{\ell_+}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log[1-\mathcal{X}_e(\zeta')^q]\right.$$
$$\left.-\frac{iq}{4\pi}\int_{\ell_-}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log[1-\mathcal{X}_e(\zeta')^{-q}]\right]$$

- Note resemblance with TBA equations

Good asymptotics

- x_m picked to satisfy $\log x_m = R/\zeta\, a_D + \ldots$ at small ζ
- Price: discontinuity
  - clockwise at $R/\zeta\, a<0$: $x_m \Rightarrow x_m\,(1-x_e)$
  - clockwise at $R/\zeta\, a>0$: $x_m \Rightarrow x_m\,(1-x_e^{-1})^{-1}$
- Same as KS factors for particle, antiparticle

General conjecture

If you had electron AND monopole

- $\log x_e = R/\zeta\, a + i\, t + R\, \zeta\, a^* - k \otimes_m \log(1 - x_m) + k \otimes_{-m} \log(1 - x_m^{-1})$
- $\log x_m = R/\zeta\, a_D + i\, t_D + R\, \zeta\, a_D^* + k \otimes_e \log(1 - x_e) - k \otimes_{-e} \log(1 - x_e^{-1})$
- $\otimes_q$ is convolution along $R/\zeta\ Z[q] < 0$

For generic BPS spectrum include all particles

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{\rm sf}(\zeta) \exp\left[-\frac{1}{4\pi i} \sum_{\gamma'} \Omega(\gamma'; u) \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 - \sigma(\gamma') \mathcal{X}_{\gamma'}(\zeta')) \right]$$

General conjecture

Good asymptotics is important!

- $\log x_q = R/\zeta\, Z[q] + \ldots..$
- Discontinuity $K_q^{\Omega[q]}$ across $R/\zeta\, Z[q]<0$
  - Compatible with asymptotics

Recovering the metric

- K_q preserves $d\log x_e \wedge d\log x_m$
- $d\log x_e \wedge d\log x_m = \omega^+/\zeta + \omega^3 + \omega^- \zeta$
- hyperkahler forms ω determine metric

Wallcrossing and hyperkahler metrics

Continuous discontinuities

- At $Z(q,u)/\zeta < 0$ discontinuity $K_q^{\Omega[q]}$
- As wall is crossed in u, lines merge and exchange
- Overall discontinuity is ordered product of $K_q^{\Omega[q]}$
- KS product!

Wall crossing formula: product is continuous

- integral equation is continuous
- solutions will be continuous
- metric will be continuous

Differential equation and isomonodromy

$$\partial_{u^j}\mathcal{X} = \left(\frac{1}{\zeta}\mathcal{A}^{(-1)}_{u^j} + \mathcal{A}^{(0)}_{u^j}\right)\mathcal{X},$$

$$\partial_{\bar{u}^{\bar{j}}}\mathcal{X} = \left(\mathcal{A}^{(0)}_{\bar{u}^{\bar{j}}} + \zeta\mathcal{A}^{(1)}_{\bar{u}^{\bar{j}}}\right)\mathcal{X},$$

$$\Lambda\partial_{\Lambda}\mathcal{X} = \left(\frac{1}{\zeta}\mathcal{A}^{(-1)}_{\Lambda} + \mathcal{A}^{(0)}_{\Lambda}\right)\mathcal{X},$$

$$\bar{\Lambda}\partial_{\bar{\Lambda}}\mathcal{X} = \left(\mathcal{A}^{(0)}_{\bar{\Lambda}} + \zeta\mathcal{A}^{(1)}_{\bar{\Lambda}}\right)\mathcal{X},$$

$$R\partial_{R}\mathcal{X} = \left(\frac{1}{\zeta}\mathcal{A}^{(-1)}_{R} + \mathcal{A}^{(0)}_{R} + \zeta\mathcal{A}^{(1)}_{R}\right)\mathcal{X},$$

$$\zeta\partial_{\zeta}\mathcal{X} = \left(\frac{1}{\zeta}\mathcal{A}^{(-1)}_{\zeta} + \mathcal{A}^{(0)}_{\zeta} + \zeta\mathcal{A}^{(1)}_{\zeta}\right)\mathcal{X}.$$

Compatible differential operators in the angles t, t_D etc.

Hitchin equations

Equations for SU(2) connection A, adjoint (1,0)-form φ

- Flat $\mathcal{A}[\zeta] = R/\zeta\, \varphi + A + R\, \zeta\, \varphi^*$
- Moduli spaces of solutions are hyperkahler
- Monodromy data of $\mathcal{A}[\zeta]$ are holomorphic functions at ζ
  - Examples: monodromies along some fixed paths

Monodromy data $M_i[\zeta]$ for fixed φ,A

- interesting function of ζ
  - Can we compute it without solving Hitchin equations?
- The hk metric is easy to compute from $M_i[\zeta]$!

A simplified example

Holomorphic Schroedinger equation

- $[h^2 d^2 - V(w)]\ F(w)=0$
- Polynomial potential $V(w)=w^k+\ldots.$

Large w behavior

- $\log F(w) \sim \pm\, w^{k/2+1}/h + \ldots.$

- Generic solution grows exponentially
- On each ray there is unique exponentially decreasing F

Stokes data

Standard definition of Stokes data

- k+2 Stokes sectors V_i
  - $\mathrm{Re}[w^{k/2+1}/h]>0$ or $\mathrm{Re}[w^{k/2+1}/h]<0$
  - Unique solution $f_i(w)$ asymptotically small in V_i
- $f_i(w)$ grows in V_{i+1} and V_{i-1}
  - $f_{i+1} - f_{i-1} = s_i[h]\, f_i$
  - $s_i[h]$ is scattering data
  - $W[f_i, f_{i+1}]=1$
  - $s_i= W[f_{i+1}, f_{i-1}]$

Easy example

Linear potential $V(w)=w$

- Too easy: $s_1 = s_2 = s_3 = i$

Quadratic potential $V(w) = w^2 - 2m$

- $\log F(w) \sim 1/2/h\, w^2 - m/h \log w$
  - $f_4 = \exp[2 \pi i\, m/h]\, f_0$ etc.
  - $s_3 = -\exp[2 \pi i\, m/h]\, s_1$ etc
  - $s_1 s_2 = -1-\exp[-2 \pi i\, m/h]$

What are small h asymptotics of $s_i[h]$?

- WKB analysis!

WKB analysis

WKB asymptotic expansion

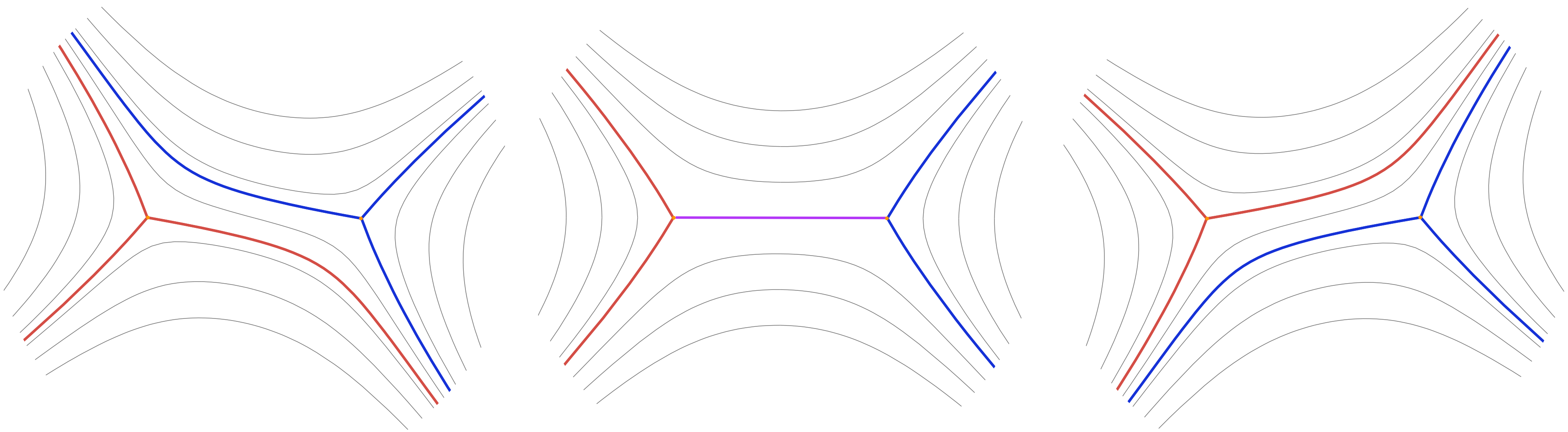
- $\log f = S_0/h + S_1 + \ldots.$ $(dS_0)^2 = V$
- Integrate phase S_0 along a path

Approximation is good or bad depending on path

- Good if $\mathrm{Re}[dS_0/h]>0$ always along path
- Can we find good paths for $s_i = W[f_{i+1}, f_{i-1}]$?
  - If so, $\log s_i = Z_i/h + \ldots$ is true

Let's look at good paths

WKB lines



$\vartheta < \vartheta_c$ $\vartheta = \vartheta_c$ $\vartheta > \vartheta_c$

Setting up a RH problem

Upper half $\log x_e = 2\pi i\, m/h$ $\log x_m = -\log s_1$

Lower half $\log x_e = 2\pi i\, m/h$ $\log x_m = \log s_2$

- Discontinuities: KS factors!
  - clockwise at $i\, m/h<0$ $x_m \Rightarrow x_m (1+x_e)$
  - clockwise at $i\, m/h>0$ $x_m \Rightarrow x_m (1+x_e^{-1})^{-1}$

Answer can be written as contour integral

- $\log x_m = m/h \log m + ...$

Comparison with exact solution

Exact answer from parabolic cylinder

$$
\begin{aligned}
s_1 &= \frac{2^{\frac{1}{2}+\frac{m}{h}} i\sqrt{\pi}}{\Gamma(\frac{1}{2}+\frac{m}{h})} \\
s_2 &= \frac{2^{\frac{1}{2}-\frac{m}{h}} i\sqrt{\pi} e^{i\pi m}}{\Gamma(\frac{1}{2}-\frac{m}{h})}
\end{aligned}
$$

For Hitchin system Tr p^2 = V(w) very similar

- singular both at z=0 and z = infinity
- For $V=w^2-2m$ same functional relations as periodic TN