Integrability from gauge fixing

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Abstract

We prove that integrable hierarchies of evolution equations can be obtained from a unique gauge theory by gauge fixing conditions. The bi - Hamiltonian structures of these hierarchies are generated by the corresponding Dirac brackets and deformations of a BRST differential.

1 Introduction

In the present work we find new formulations for the integrability conditions of Hamiltonian systems of evolution equations.

Using a variational approach, we prove that:

- any Hamiltonian evolution equations can be derived from a quantum BRST field theory. The gauge fixing fermion determines the Poisson structure and its deformations the existence of an infinite number of compatible brackets.
- Hamiltonian systems with an exact two form are derived from a singular action with second class constraints. The separation of this system of constraints into first class and gauge fixing ones can be used to prove the existence of bi-Hamiltonian and thus integrable structures.

2 Hamiltonian systems and the ghosts

In this section we show that any Hamiltonian system of evolution equations can be regarded as a classical field theory with a BRST-anti-BRST exact effective Hamiltonian.

Notations:

Let $\mathcal{F}$ be the space of functionals $\mathcal{R} = \int R dx$, where the differential functions $R(x,\phi) \equiv R[\phi]$ are defined on $M \subset X \times \Phi$, an open set of the space of independent and dependent variables $x = (x^1, ..., x^p)$ and $\phi = (\phi^1, ..., \phi^n)$

Let $\Omega : \mathcal{A}^n \rightarrow \mathcal{A}^n$ be a $n \times n$ matrix skew-adjoint differential operator which may also depend on ϕ; $\mathcal{A}$ - the algebra of differential functions $R[\phi]$ over M.

Theorem 1

The Poisson bracket defined for any two functionals in $\mathcal{F}$ by the operator Ω:

$$\{\mathcal{G}_1, \mathcal{G}_2\}_\Omega = \int \delta\mathcal{G}_1 \; \Omega \; \delta\mathcal{G}_2 dx \qquad (1)$$

can be expressed in terms of a standard BRST bracket $\{.,.\}$ and/or of a $\overline{Q}_\Omega$ - bracket, on an extended space: $\widetilde{M} = X \times \widetilde{\Phi}$, with $\widetilde{\Phi} = (\phi, \pi, C, \mathcal{P})$, where $\pi, C, \mathcal{P}$ are auxiliary variables.

$$\begin{array}{c} Q = \int dx C^a \pi_a \\ \overline{Q}_\Omega = - \int dx \left(\mathcal{P}_a \pi_b \Omega^{ba} + \frac{1}{2} \mathcal{P}_a \mathcal{P}_b C^c \delta_c \Omega^{ba} \right) \end{array} \qquad (2)$$

The conjugated variables (ϕ, π) and $(C, \mathcal{P})$ satisfy:

$$\begin{array}{c} \{\phi^a(x), \pi_b(y)\}_{x^0=y^0} = \delta^a_b \delta(x-y) \\ \{C^a(x), \mathcal{P}_b(y)\}_{x^0=y^0} = \delta^a_b \delta(x-y) \end{array} \qquad (3)$$

$$\begin{array}{c} \{\mathcal{G}_1, \mathcal{G}_2\}_{\overline{Q}_\Omega} \equiv \frac{1}{2} \left\{ \{\mathcal{G}_1, \overline{Q}_\Omega\}, \{Q, \mathcal{G}_2\} \right\} - \\ \frac{1}{2} \left\{ \{\mathcal{G}_1, Q\}, \{\overline{Q}_\Omega, \mathcal{G}_2\} \right\} \end{array} \qquad (4)$$

Consequence:

If the charge $\overline{Q}_\Omega$ defines a nilpotent differential $\overline{s}_\Omega . \equiv \{\overline{Q}_\Omega, .\}$, $\{\overline{Q}_\Omega, \overline{Q}_\Omega\} = 0$,

then: the differential operator Ω is Hamiltonian since the corresponding bracket (1) satisfies the Jacobi identity.

Note also that: the BRST charge Q in (2) defines a nilpotent differential $s \equiv \{Q, .\}$ by construction.

Theorem 2

Any system of Hamiltonian evolution equations:

$$\frac{\partial \phi^a}{\partial t} = \Omega^{ab}\ \delta_b \mathcal{H}\left[\phi\right] \tag{5}$$

with a Hamiltonian functional $\mathcal{H} = \int H dx$ is given by the equations of motion:

$$\dot{\phi}^a = \left\{\phi^a, \mathcal{H}^{eff}\right\} = \{\phi^a, \mathcal{H}\left[\phi\right]\}_{\overline{Q}_\Omega} \tag{6}$$

of a field theory having: $H^{eff} = \{Q, \{H[\phi], \overline{Q}_\Omega\}\}$

The equations of motion can be derived from $\mathcal{L}^{eff} = \int dx L^{eff}$ with:

$$L^{eff} = \pi_a\ \dot{\phi}^a + \mathcal{P}_a\ \dot{C}^a - H^{eff} \tag{7}$$

and:

$$H^{eff} = \pi_a \Omega^{ab} \delta_b H + \mathcal{P}_a C^a \delta_b \left(\Omega^{ac} \delta_c H \right) \tag{8}$$

Geometrical interpretation: $C^a :$ basis of forms, $d\phi^a$ and $\mathcal{P}_a :$ basis of vector fields.

Grading:

$$\begin{gathered} gh\left(\phi\right) = gh\left(\pi\right) = 0 \\ gh\left(C\right) = 1 \\ gh\left(\mathcal{P}\right) = -1 \\ gh\left(s\right) = 1 \\ gh\left(\overline{s}_\Omega\right) = -1 \\ gh\left(\mathcal{H}\left[\phi\right]\right) = gh\left(\mathcal{H}^{eff}\right) = 0 \end{gathered} \tag{9}$$

Consequence: a conserved functional $\mathcal{G}$ satisfies:

$$\delta\mathcal{G}\ \Omega\ \delta\mathcal{H} = 0 \tag{10}$$

$<=>$

$$\left\{\mathcal{G}, \mathcal{H}^{eff}\right\} = \left\{\mathcal{G}, \mathcal{H}\right\}_{\overline{Q}_\Omega} = 0 \tag{11}$$

In cohomological terms: $s\mathcal{G}$ belongs to the cohomology of $\overline{s}_\Omega$ at ghost number 1 (the groups of the generators of non-trivial global symmetries).

3 Bi-Hamiltonians from gauge-fixing

In this section we prove that the bi-Hamiltonian structures of an integrable system are obtained from gauge fixing invariance of the associated field theory.

Theorem 3

Let $s, \overline{s}_0, H_0$ be the symmetries and the functional defining the BRST field theory corresponding to a given PDE system.

Let the theory be non - anomalous, i.e. the cohomology groups of s and $\overline{s}_0$ are trivial for ghost numbers ≥ 1 and ≤ -1, respectively.

If a one-parameter deformation of the symmetry exists, so that:

$$\overline{s}_1 \overline{s}_0 \mathcal{H}_0 = 0 \tag{12}$$

then:

(i) the original system is bi-Hamiltonian, there exists an infinite number of conserved functionals $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_m, ...$ and

(ii) the new gauge fixing $s\overline{s}_{def}\mathcal{H}_{def}$, where $\mathcal{H}_{def}$ is a complete deformation of $\mathcal{H}_0$, does not change the equations of motion.

Remark 1: if different deformations $\overline{s}_{1a}$ of $\overline{s}_0$ exist, so that $\overline{s}_{1a}\overline{s}_0 H_0 = 0$, then one may generate different hierarchies corresponding to:

$$\overline{s}_{1a}\mathcal{H}_m = \overline{s}_0\mathcal{H}_{m+1} \tag{13}$$

Remark 2: the existence of a recursion operator R implies:

$$\left\{\overline{Q}_{\Omega_k}, \mathcal{F}\right\} = \left\{\overline{Q}_{R^k\Omega_0}, \mathcal{F}\right\} \tag{14}$$

Although each of the pairs $\overline{Q}_m, \mathcal{H}_{n-m}$ corresponds to different effective Hamiltonians:

$\mathcal{H}_n^{eff} = -\{Q, \psi_n\} = -\left\{Q, \left\{\overline{Q}_{\Omega_m}, \mathcal{H}_{n-m}\right\}\right\}$ and different equations of motions, the total deformed gauge-fixing term corresponds to the original path integral.

4 Examples

We give a few simple but celebrated examples of integrable hierarchies and their anti-BRST transformations.

From: gauge fixing term $s\bar{s}_\alpha H_a$: $\Omega_k = d\phi^a \wedge \phi^b \left(\Omega_k\right)_{ab}$ (for $k = \alpha, \beta, \gamma$ and $a, b = 1, 2$), with:

$$\Omega_k^{11} = -\Omega_k^{22} = 0 \qquad (15)$$

for $k = \alpha, \beta$ and:

$$\Omega_\beta^{12} = -\Omega_\beta^{21} = -T_S \qquad (16)$$

where T_S is the Schroedinger operator and

$$\Omega_\alpha^{12} = -\Omega_\alpha^{21} = -1 \qquad (17)$$

$$\Omega_\gamma^{ab} = T^{ab} \qquad (18)$$

where T is differently defined (as T_{KdV}, or T_{NLS} or T_{sG}) for (complex) KdV, non linear Schroedinger or sine-Gordon hierarchies, respectively.

For the non - linear Schroedinger hierarchy: $T^{11} = -\partial + 2\phi^2 D^{-1}\phi^2$, $T^{22} = -\partial + 2\phi^1 D^{-1}\phi^1$, $T^{12} = -2\phi^2 D^{-1}\phi^1$, $T^{21} = -2\phi^1 D^{-1}\phi^2$, with $D^{-1} = \frac{1}{2}(\int^x + \int_x)$.

linear Schrödinger

$\uparrow \bar{s}_\beta$

$\frac{\partial\psi}{\partial t} = -i\psi$ (from $\bar{s}_0 = \bar{s}_\alpha$)

$\downarrow \bar{s}_\gamma$

$\frac{\partial\psi}{\partial t} = \partial_x\psi$

$\downarrow \bar{s}_\gamma$

$\frac{\partial\psi}{\partial t} = i\,(\partial_{xx}\psi + |\psi|^2\psi)$

$\downarrow \bar{s}_\gamma$

higher NL Schrödinger

5 Integrability without ghosts: the role of constraints

In this section we show that when the Poisson two form Ω is exact, the Dirac analysis of constraints gives us the conditions for a second Hamiltonian operator to exist.

Property 1

The variational principles for the following actions are equivalent [1]:

(i)

$$S_e = \int dt(\pi\dot{\phi} - H(\pi, \phi, t) - \lambda_a \chi_a) \qquad (19)$$

where $\chi_a(\pi, \phi) = 0$ are second class constraints.

(ii)

$$S = \int dt(A\dot{\eta} - h(\eta)) \qquad (20)$$

where we denote by η all the fields on the reduced phase space.

The reason is that the inverse Ω of $\partial_i A_j - \partial_j A_i$ coincides with the two form induced by the Dirac bracket on the constraint surface.

Consequences

(c1) A bi-Hamiltonian system $\dot{\eta} = (\Omega_0)_{\eta\eta'}\delta_{\eta'}h_1 = (\Omega_1)_{\eta\eta'}\delta_{\eta'}h_0$ is obtained from a unique, singular action $S = \int dtL(\phi, \dot{\phi}, t)$ which in turn can be written as (19).

(c2) Since the system of second class constraints χ_a can be transformed into a system of first class constraints ϕ_1^n plus gauge fixing constraints ϕ_2^n, the integrability, by Magri's theorem, is translated into gauge fixing invariance.

(c3) the integrability conditions are equivalent to the fact that there exist infinitely many choices of the gauge fixing constraints $\phi_2^{n(k)}$, so that: $\{h_k, h_j\}_{\Omega_\alpha} = 0$ and $\Omega_\alpha \delta h_k = \Omega_\beta \delta h_{\alpha+k-\beta}$.

Here, $h_k = H|_{\chi=0}$ for the system of constraints χ_a with ϕ_1^n and a choice k of the fixing $\phi_d^{n(k)}$ and Ω_α is the inverse of the matrix of constraints $\{\chi_i, \chi_j\}$ for a specific choice $\phi_d^{n(k)}$. The field configurations are related to each other by a D-transformation [2].

6 No ghost examples

In this section we give a few examples of bi-Hamiltonian structures which arise as a result of changing the gauge fixation.

Example 1 The KdV hierarchy

We can read the two form and the Hamiltonian h of the reduced action directly from a first order action for the potential $U_x = u$, which is indeed of the type (20):

$$S = \int dt(\frac{1}{2}U_t U_x - (U_x^3 + \frac{1}{2}U_{xx}^2)) \qquad (21)$$

We find: $h = -U_x^3 + \frac{1}{2}U_{xx}^2$ and $O(x,y) = \frac{\delta}{\delta U(y)}U_x(x) - \frac{\delta}{\delta U(x)}U_y(y) = \delta_y(x-y) - \delta_x(y-x)$. Its inverse is $\theta(x-y)$ and gives, in terms of u, the well known Hamiltonian operator $\frac{d}{dx}$. The constraints here are of the type $\chi = \pi_U - f(U)$.

Example 2 Duality invariant systems

The reduced phase space actions of free electromagnetism or linearized gravity are of the type

$S = \int dt(a\dot{\eta} - h(\eta))$. For free electromagnetism we have [5] $\eta = A_i^{Ta}$, $h_0 = \int d^3x \frac{1}{2} f^2(A)$ and $a_i \sim f_i(A)$ for an infinite number of choices of the functions f_i, related to each other by Ω and the recursion operator, which all satisfy (c3). Here, too, the constraints are of the type $\chi = \pi_A - f(A)$.

7 Conclusions

In this work, we expressed the bi-Hamiltonian integrability conditions as gauge fixing invariance in two different ways. The former translates into cohomological conditions for the BRST differential. The latter translates into constraints' analysis.

Appendix: *proof of Theorem 3*

We denote $\Lambda_0 = \overline{s}_0 \mathcal{H}_0$ Then equation (12) becomes: $\overline{s}_1 \Lambda_0 = 0$ and we can write $\Lambda_1 = \overline{s}_1 \mathcal{H}_0$ which implies:

$$\overline{s}_0\Lambda_1 = \overline{s}_0\overline{s}_1\mathcal{H}_0 = -\overline{s}_1\overline{s}_0\mathcal{H}_0 = 0 \tag{22}$$

If the cohomology of $\overline{s}_0$ is trivial at ghost number -1, thus any $\overline{s}_0$ - closed form is exact, then it exists a $\mathcal{H}_1$ so that $\Lambda_1 = \overline{s}_0\mathcal{H}_1$ Moreover, the functional $\mathcal{H}_1$ is conserved:

$$\left\{\mathcal{H}_1, \mathcal{H}^{eff}\right\} = \{\mathcal{H}_1, \mathcal{H}_0\}_{\overline{Q}_{\Omega_0}} = \{\mathcal{H}_1, \mathcal{H}_0\}_{\overline{Q}_{\Omega_1}} = 0 \tag{23}$$

One continues then to construct $\Lambda_2 = \overline{s}_1\mathcal{H}_1$ and apply the same argument to prove that a term $\mathcal{H}_2$ exists. After m steps one obtains a whole set of equalities of the form:

$$\Lambda_{m+1} = \overline{s}_1\mathcal{H}_m = \overline{s}_0\mathcal{H}_{m-1} \tag{24}$$

In order to prove the second part of the theorem, we note that the deformed BRST-antiBRST symmetry is given by $s_{def} = s + \overline{s}_{def}$ where:

$$\overline{s}_{def} = \overline{s}_0 - \lambda^1\overline{s}_1 \tag{25}$$

and the complete gauge-fixing term is $s\overline{s}_{def}\mathcal{H}_{def}$ with $\mathcal{H}_{def}=\mathcal{H}_0+\lambda\mathcal{H}_1+\lambda^2\mathcal{H}_2+\lambda^3\mathcal{H}_3+...$, for real λ. The invariance of the path integral reduces, modulo s-exact terms, to $\left(\overline{s}_0-\lambda\overline{s}_1\right)\mathcal{H}_{def}=\overline{s}_0\mathcal{H}_0$. Then:

$$\overline{s}_0\left(\mathcal{H}_{def}-\mathcal{H}_0\right)=\left(\lambda\overline{s}_1\right)\mathcal{H}_{def} \tag{26}$$

$$\overline{s}_1\overline{s}_0\mathcal{H}_{def}=0 \tag{27}$$

and includes the "initial conditions" (12). Using the nilpotency of the $\overline{s}_{def}$ differential, $\overline{s}_0\overline{s}_1+\overline{s}_1\overline{s}_0=0$ and (24), obtains (ii). q.e.d.

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