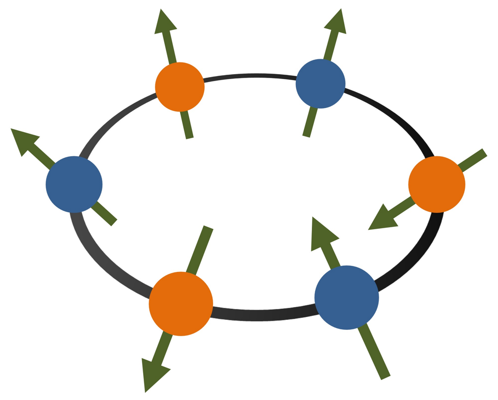

The Morphology of $N=6$ Chern-Simons Theory

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1. Introduction

1.1 ABJM Theory & Integrability

The conformal $\mathcal{N}=6$ supersymmetric Chern-Simons matter theory proposed by Aharony, Bergman, Jafferis and Maldacena in order to describe the worldvolume of M2-branes has been the subject of intense study since its inception in 2008. It is a 3-dimensional gauge theory with gauge group $U(N)\times U(N)$ and superconformal group $OSp(6|4)\supset SO(6)\times Sp(4,\mathbb{R})$.

The fundamental fields of ABJM theory are four complex scalars and their fermionic partners which transform in the $(N,\overline{N})$ of the gauge group and in the so-called 'singleton', representation of the superconformal group, denoted $\mathcal{V}^1$. In addition we have their complex conjugate fields in the $(\overline{N},N)$ and conjugate singleton, $\overline{\mathcal{V}}^1$, representation of the gauge and superconformal groups respectively, as well as non-dynamical gauge fields with Chern-Simons levels $+k$ and $-k$. The theory has a 't Hooft limit where $N,k\to\infty$ with $\lambda=N/k$ fixed, in which it admits a dual description in terms of type IIA string theory on $AdS_4\times CP^3$.

Integrability plays a key role in unveiling the structure of planar ABJM theory. At leading order in the weak coupling expansion, Minahan and Zarembo, 2008, Gaiotto, Giombi and Yin, 2008, and Bak and Rey, 2008, proved that the asymptotic spectrum of anomalous dimensions for certain subsets of operators of the theory is encoded in an integrable spin chain Hamiltonian, of the general form

$$\Delta_2=\lambda^2\sum_{i=1}^{2L}(D_2)_{i,i+1,i+2}$$

where the Hamiltonian density D_2 acts simultaneously on three adjacent sites, and $2L$ is the total number of sites.

This result was recently extended with the construction of the full 2-loop Dilatation operator by Zwiebel, 2008, and Minahan, Schulgin and Zarembo, 2008, and moreover an exact magnon S-matrix for this spin chain has been proposed by Gromov and Vieira.

The group-theoretic results we present here aim to provide some groundwork for further spectroscopic analysis of the ABJM theory and the determination of its partition function on $R\times S^2$, based on the aforementioned work.

1.2 $OSp(6|4)$ Representations & super-Young Tableaux

We will employ the oscillator method, first applied to $OSp(2m|2n)$ groups by Gunaydin and Hyun, 1988, in order to construct the irreducible multiplets under which the states of the ABJM spin chain, equivalently the single trace operators of the dynamical fields of the theory (and their derivatives), transform.

For a multiplet containing f fields, we start by defining $f+f$ superoscillators in the covariant and contravariant representations of the $U(2|3)$ subsupergroup respectively, and then we express the $OSp(6|4)$ generators as their bilinears. By construction, the generators naturally split into the $U(2|3)$ generators $M^A{}_B$, and those that either increase, S^{AB}, or decrease, S_{AB}, the central $U(1)$ charge of $U(2|3)$. In order to build irreducible multiplets, we look for the so called Lowest Weight States (LWS), which are annihilated by the S_{AB}. Once such a state has been found, the entire multiplet is built by successively acting on it with the S^{AB} generators.

In this manner, a natural labeling of the multiplet is in terms of its length f and the $U(2|3)$ super-Young tableau (SYT) of its LWS, which encodes how many superoscillators exist in the LWS, and whether they are graded symmetric or antisymmetric when we exchange the indices between any two of them.

A representation of length f whose SYT has n rows with $(k_1,k_2,\ldots k_n)$ boxes in the (first, second,...,n-th) row will be denoted as $\mathcal{V}^f_{k_1,k_2,\ldots,k_n}$. More conventionally $\mathcal{V}^f_{k_1,k_2,\ldots,k_n}$ is labeled by its Cartan charges $[\Delta,j;d_1,d_2,d_3]$ together with f, where (Δ,j) is the conformal group $SO(3,2)$ scaling dimension and spin, and $[d_1,d_2,d_3]$ are the Dynkin labels of the $SU(4)$ R-symmetry group. We have proven that the two different labelings are related by

$$(\Delta,j)=\Big(\frac{1}{2}\left(\max(k_1-3,0)+\max(k_2-3,0)+f\right),\frac{1}{2}(\max(k_1,3)-\max(k_2,3))\Big)\,,$$

$$[d_1,d_2,d_3]=[f-\sum_{i=1}^{n}\min(k_i,2),\sum_{i=1}^{n}\delta_{k_i,2},\sum_{i=1}^{n}\delta_{k_i,1}]\,.$$

2. Group-theoretic Results

2.1 Characters

Characters are important quantities because the encode the weight content of all states within a multiplet. The oscillator method can calculate them in full generality, as F.A. Dolan, 2008, has done using a different set of tools, but for simplicity here we'll focus on the 1-parameter family of characters

$$V^f_{k_1,\ldots,k_n}(x)=\mathrm{Tr}_{\mathcal{V}^f_{k_1,\ldots,k_n}}[x^\Delta].$$

For all representations up to $f=2$ we have

$$\begin{aligned}
V^1(x)=V^1_1(x)&=\frac{4\sqrt{x}}{(1-\sqrt{x})^2}\,,\\
V^2(x)=V^2_{1,1}(x)&=\frac{2x(5-x)}{(1-\sqrt{x})^3}\,,\\
V^2_1(x)&=\frac{x(15+7\sqrt{x}-3x-3x^{\frac{3}{2}})}{(1-\sqrt{x})^3}\,,\\
V^2_k(x)&=x^{\frac{k-1}{2}}\frac{(1+\sqrt{x})^3}{(1-\sqrt{x})^3}\left(k-2+6\sqrt{x}-(k+2)x\right)\qquad k\geq 2\,,
\end{aligned}$$

where representations that have the same partition functions are conjugate to each other. Length-3 and 4 partition functions have been obtained in a similar fashion.

2.2 Tensor Product decompositions

Polyakov, 2001, realized that there is a very beautiful and useful analogy between the counting of local gauge-invariant operators in gauge theories such as SYM and linguistics. The elementary fields (and their derivatives) are thought of as 'letters' which are strung together inside single trace operators as 'words', products of which can then be thought of as 'sentences'.

Since the ABJM elementary fields transform in either the $(N,\overline{N})$ or the $(\overline{N},N)$ bi-fundamental representations they must appear in alternating order inside any single-trace operator. If we wish to extend the linguistic analogy to this case we could perhaps say that the ABJM alphabet is divided into consonants and vowels, comprising respectively the two $OSp(6|4)$ singleton representations $\mathcal{V}^1$ and $\overline{\mathcal{V}}^1$. Every word in the ABJM language has even length and consists of alternating vowels and consonants.

Taking this analogy further, we can think of the irreducible representations arising in the tensor product of two singletons as the 'digraphs' of the ABJM language, groups of two successive letters whose phonetic value is a distinct sound, such as *aw* in *saw*. Similarly, we can refer to the multiplets appearing in triple singleton products, on which the Hamiltonian density acts, as 'syllables', being the building blocks of words. We have found the digraphs and syllables of interest to be

$$\mathcal{V}^1\otimes\overline{\mathcal{V}}^1=\sum_{m=0}^{\infty}\mathcal{V}^2_{2m+1}\,,\qquad \mathcal{V}^1\otimes\overline{\mathcal{V}}^1\otimes\mathcal{V}^1=\sum_{m=0}^{\infty}(m+1)\left(\mathcal{V}^3_{2m+1}+\mathcal{V}^3_{2m+2,1}\right).$$

In the same fashion, we have moved on to determine the four-letter words of the ABJM language, as they correspond to the shortest length operators whose energies we can determine with the R-matrix construction of the spin chain Hamiltonian. Focusing on the physical, cyclically invariant states, we have

$$\begin{aligned}
(\mathcal{V}^1\otimes\overline{\mathcal{V}}^1)^2_+=&\sum_{j=1}^{\infty}j(j+1)(\mathcal{V}^4_{4j}+\overline{\mathcal{V}}^4_{4j}+\mathcal{V}^4_{4j+2}+\overline{\mathcal{V}}^4_{4j+2})+[2j(j-1)+1]\mathcal{V}^4_{4j-3,1}\\
&+\sum_{j=1}^{\infty}\sum_{p=1}^{j}\Big\{2\left[j(j+1)-p^2\right](\mathcal{V}^4_{4j,4p}+\mathcal{V}^4_{4j+2,4p})\\
&\qquad+2\left[j^2-p(p-1)\right](\mathcal{V}^4_{4j-1,4p-1}+\mathcal{V}^4_{4j-1,4p-3})\\
&\qquad+\left[2j^2-1-2p(p-1)\right](\mathcal{V}^4_{4j-2,4p-2}+\mathcal{V}^4_{4j,4p-2})\\
&\qquad+\left[2j(j+1)+1-2p^2\right](\mathcal{V}^4_{4j+1,4p-1}+\mathcal{V}^4_{4j+1,4p+1})\Big\}\,.
\end{aligned}$$

3. Applications on the Two-Loop Dilatation Operator

3.1 The Trace $\langle D_2(x)\rangle$ of the Hamiltonian Density

Here we calculate the trace $\langle D_2(x)\rangle=\mathrm{Tr}_{\mathcal{V}^1\otimes\overline{\mathcal{V}}^1\otimes\mathcal{V}^1}[x^\Delta D_2]$ of the Hamiltonian density, whose importance lies in the fact that it enters into the formula for the two-loop correction to the partition function of planar ABJM theory on S^2.

We will need the results of Section 2.1, together with Zwiebel's explicit form of the Hamiltonian density,

$$(D_2)_{123}=\sum_{j=0}^{\infty}h(j)\mathcal{P}^{(j)}_{12}+\sum_{j_1,j_2,j_3=0}^{\infty}(-1)^{j_1+j_3}\big(\tfrac{1}{2}h(j_2-\tfrac{1}{2})+\ln 2\big)\big(\mathcal{P}^{(j_1)}_{12}\mathcal{P}^{(j_2-\frac{1}{2})}_{13}\mathcal{P}^{(j_3)}_{12}+\mathcal{P}^{(j_1)}_{23}\mathcal{P}^{(j_2-\frac{1}{2})}_{13}\mathcal{P}^{(j_3)}_{23}\big),$$

where $h(j)$ are harmonic numbers and $\mathcal{P}^{(j)}_{ab}$ is the projection operator whose image is spanned by states with $OSp(6|4)$ spin j in the tensor product space of sites a and b.

Due to symmetry, the trace calculation reduces to a sum of characters with harmonic number coefficients, which finally yields

$$\langle D_2(x)\rangle=8\sqrt{x}\frac{(1+\sqrt{x})^2}{(1-\sqrt{x})^6}\left[\sqrt{x}+x+(1-6\sqrt{x}+x)\log(1-\sqrt{x})\right]\,.$$

3.2 The Two-Loop Hagedorn Temperature

It is known that, like planar SYM on S^3, ABJM on S^2 has a Hagedorn temperature $T_{\rm H}$, namely for $T>T_{\rm H}$ the partition function of the theory becomes divergent.

In the 't Hooft limit $T_{\rm H}$ is a nontrivial function of λ and at zeroth order it's easy to show that it's obtained from $x_{\rm H}=17-12\sqrt{2}$, the smallest solution of the equation $V^1(x)=1$, where $V^1(x)$ is the singleton partition function and the relation between the variable x and temperature T is always given by $x=e^{-1/T}$.

The two-loop correction $\delta T_{\rm H}$ is in turn related to the pole structure of the two-loop partition function, and following the general analysis of Spradlin and Volovich,2004, we deduce that

$$\frac{\delta T_{\rm H}}{T_{\rm H}}=\frac{\lambda^2}{\sqrt{2}}\langle D_2(x_{\rm H})\rangle=2\lambda^2(\sqrt{2}-1)\,.$$

3.3 Spectrum of Low-Lying States

We use the results of Section 2.2 and Zwiebel's explicit form for the planar 2-loop Dilatation operator to obtain the corresponding spectrum of all length-4 primary states with $\Delta\leq 6$ in the $OSp(4|2)$ sector, as shown in the table below.

Δ	$SO(4)$	SYT	D_2	
2	$[2,2]$	$(1,1)$	0	
	$[0,0]$	$(2,2)$	8	
2.5	$[1,1]$	$(3,1)$	6^2	
3	$[2,0]$	(4)	6^2	+ conj.
	$[0,0]$	$(4,2)$	8	
3.5	$[1,1]$	$(5,1)$	$4,8^2,11^2$	
4	$[2,0]$	(6)	10^2	+ conj.
	$[0,0]$	$(6,2)$	$(25/3)^4,32/3,12^2$	
4.5	$[1,1]$	$(7,1)$	$(7.94)^2,(134/15)^4,(14.06)^2$	
5	$[2,0]$	(8)	$(7.87)^2,(10.76)^2,(13.36)^2$	+ conj.
	$[0,0]$	$(8,2)$	$32/3,12^2,(63/5)^4$	
5.5	$[1,1]$	$(9,1)$	$6,(28/3)^2,(247/21)^4,(12.28)^2,(40/3)^2,(16.22)^2$	
6	$[2,0]$	(10)	$(11.25)^2,(79/7)^2,(15.75)^2$	+ conj.
	$[0,0]$	$(10,2)$	$(9.62)^4,(11.03)^4,184/15,(40/3)^2,(46/3)^2,15.38^4$	

The SYT column refers to the super-Young tableau labeling of the states. Superscripts denote the multiplicities of eigenvalues, when they are larger than one. The label '+conj.' represents conjugate states with $SO(4)$ labels reversed.