"Integrability in Gauge and String Theory", AEI Potsdam, 30 July 2009

Y/T-systems and full spectrum of planar AdS/CFT

Vladimir Kazakov (ENS,Paris)

with N.Gromov and P.Vieira, arXiv:0812.5091

arXiv:0901.3753

arXiv:0906.4240

with N.Gromov, A.Kozak and P.Vieira, arXiv:0902.4458

Outline

- Let us solve exactly a 4-dimensional Yang-Mills theory!
  QCD is difficult. Try N=4 SYM (supersymmetry often helps).

- Remarkable progress due to the AdS/CFT correspondence (in the last 12 years) and due to integrability for planar SYM (last 6 years).

- Until recently, we knew only the anomalous dimensions at any coupling of asymptotically long operators from asymptotic Bethe ansatz (ABA).

  Minahan,Zarembo'03
  ....many efforts, many people...
  Beisert,Eden, Staudacher'06

- Our result: we conjecture an efficient set of equations, Y-system, for the spectrum of anomalous dimensions of ALL operators for ALL couplings.

- My talk: Generalization of ABA to finite size operators by "standard" techniques in integrable models: functional Y-system (and its ABA limit).

- Kolya's talk: integral Y-system and numerical calculation of Konishi dimension in a wide range of couplings, from week to strong regime.

N=4 SYM and string in AdS$_5$ x S^5

$$\mathcal{S}_{SYM} = \frac{1}{\lambda}\int d^4x, \mathrm{Tr}\left(F^2 + (\mathcal{D}\Phi)^2 + \bar{\Psi}\mathcal{D}\Psi + \bar{\Psi}\Phi\Psi + [\Phi,\Phi]^2\right)$$

$$\longleftrightarrow$$

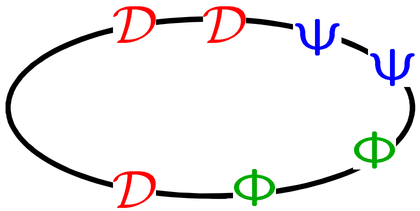
$$S_{sigma} = \frac{\sqrt{\lambda}}{4\pi}\int d\tau \int_0^L d\sigma \left((\partial x)^2 + \Lambda(x^2-1) + \text{fermions}\right)$$

Metsaev-Tseytlin superstring sigma model

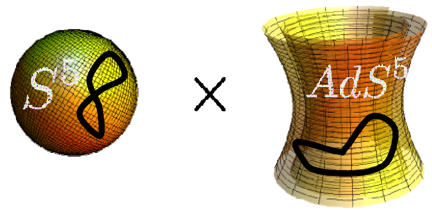
Maldacena'97
Gubser,Klebanov,Polyakov'98
Witten'98

AdS/CFT duality

$$\mathcal{O}(x) = \mathrm{Tr}\left[\mathcal{D}\mathcal{D}\Psi\Psi\Phi\Phi\mathcal{D}\Psi\ldots\right](x) \quad \longleftrightarrow$$



$$\lambda = g_{YM}^2 N = \frac{R^4}{\alpha'^2} = g_s N$$



$$\langle \mathcal{O}_A(x)\mathcal{O}_B(0)\rangle = \frac{\delta_{AB}}{|x|^{2\Delta_A(\lambda)}}$$

Dimension of renormalized local operator = Energy of string state

Global superconformal symmetry $\rightarrow$ psu(2,2|4) $\leftarrow$ isometry of background

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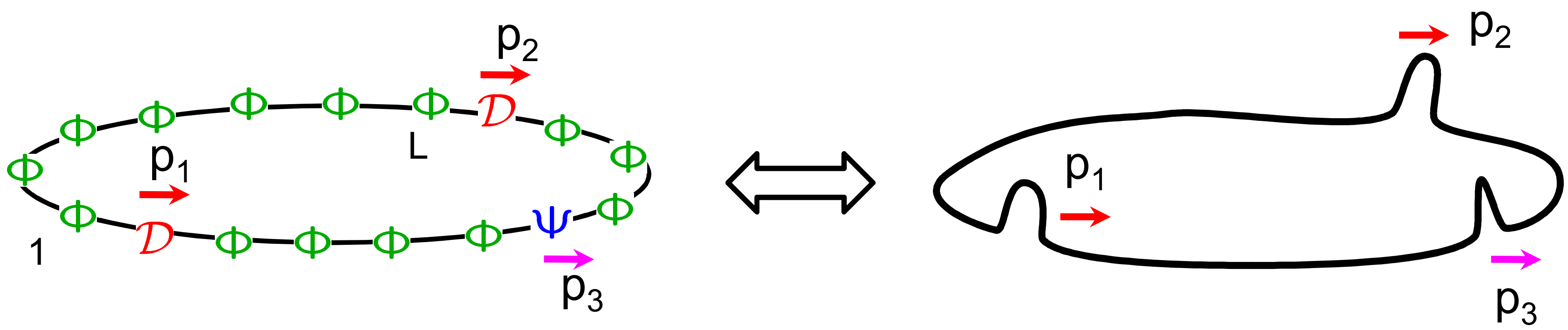
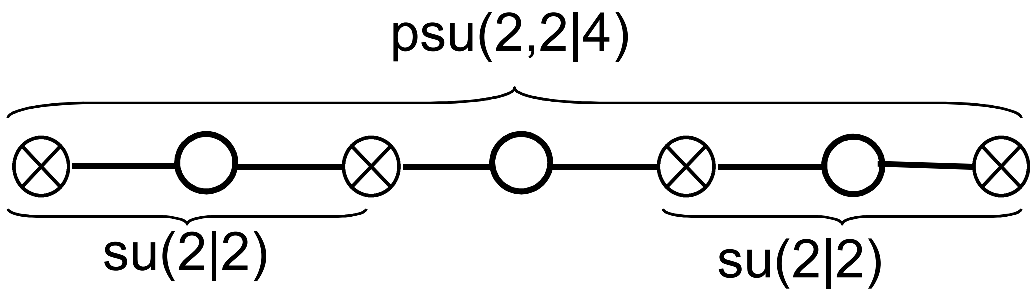
Integrability and S-matrix



World-sheet light-cone gauge breaks the superconformal symmetry:

Staudacher'04
Beisert'05
Janik'06
Beisert,Hernandez
Lopez'06
Beisert,Eden
Staudacher'06

$$psu(2,2|4) \Longrightarrow su(2|2) \times su(2|2)$$



• S-matrix of AdS/CFT from Zamolodchikov bootstrap:

$$S_{PSU(2,2|4)}(p_1,p_2) = S_0^2(p_1,p_2)\ S_{SU(2|2)}(p_1,p_2) \times S_{SU(2|2)}(p_1,p_2)$$

• Asymptotic integrability: factorized scattering, asymptotic Bethe ansats…
• Strictly speaking, valid only for long operators/high $AdS_5 \times S^5$ charges

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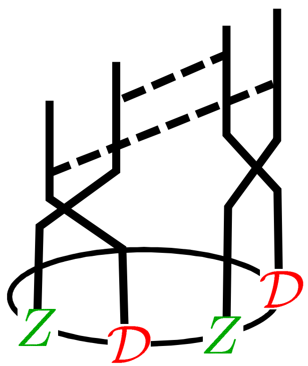
Finite size operators and wrapping

- Finite length L $\leftrightarrow$ "short" operators: $\mathcal{O}_{Konishi} = \text{Tr}\ [\mathcal{D}, Z]^2$

Fiamberti,Santambroglio, Sieg,Zanon'08,Velizhanin'08

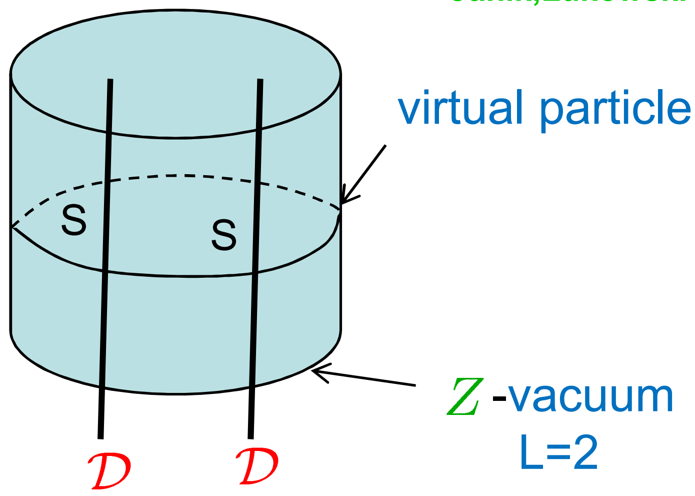
Planar graphs (4 loops)



$+ \ldots\ \Longleftrightarrow$

Lüscher corrections for string: (5 loops)

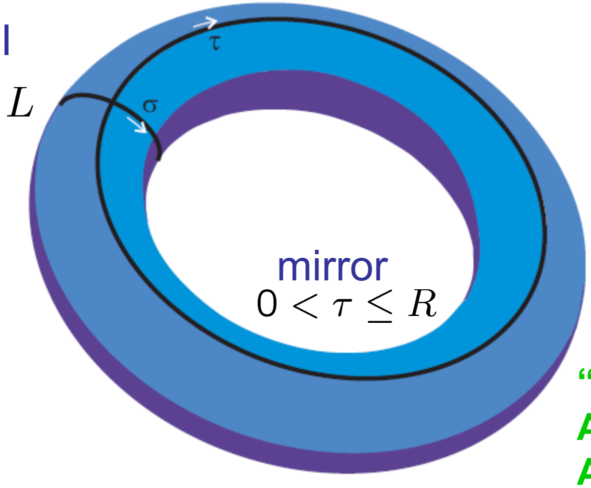
Bajnok,Janik'08
Bajnok,Janik,Lukowski'09
Bajnok,Hegedus, Janik,Lukowski'09



- Our Y-system confirms the 4-loop results! It works for all operators at any coupling λ

TBA: Free energy = ground state $E_0(L)$

Al.Zamolodchikov'90



"mirror" original

$$\mathcal{Z} = \sum_n e^{-E_n(R)L} = \sum_n e^{-E_n(L)R}$$

$$\downarrow R=\infty$$

$$e^{-E_0(L)R}$$

"mirror" for AdS5xS5 string:
Ambjorn,Janik,Kristjansen'06
Arutyunov,Frolov'07,'08,'09

- I.e. from the asymptotic spectrum (R=∞) we can compute the ground state energy for ANY finite length L using "string hypothesis" (Takahashi bound states: su(2|2) spin chain is similar to Hubbard model)

N.Dorey'06, Beisert'06. In "mirror" theory: Arutyunov,Frolov'09

- Functional Y-system Gromov,V.K.,Kozak,Vieira'09
- In this way, AdS/CFT TBA-type eqs for the ground state were found

Bombardelli,Fioravanti,Tateo'09
Gromov,V.K.,Kozak,Vieira'09
Arutyunov,Frolov'09

- Excited states can be included by a certain analytical continuation

Bazhanov,Lukyanov,A.Zamolodchikov'96
P.Dorey,Tateo'96
Fioravanti,Mariottini,Quattrini,Ravanini'96

- Integral eqs for excited states in AdS/CFT

Gromov,V.K.,Kozak,Vieira'09

“Toy” model: $SU(2)_L$ x $SU(2)_R$ principal chiral field (PCF)

$$\mathcal{L} = \frac{\sqrt{\lambda}}{4\pi}\int dt \int_0^L dx \, \mathrm{tr}\, \left(g^{-1}\partial_\mu g(x,t)\right)^2, \qquad g \in SU(2).$$

- Asymptotically free theory with dynamically generated mass $m = \Lambda\, e^{-\frac{\sqrt{\lambda}}{4\pi}}$

- S-matrix: Satisfies Yang-Baxter, unitarity, analyticity and crossing ($L \to \infty$)

$$\begin{aligned} E &= m \cosh \pi\theta \\ P &= m \sinh \pi\theta \end{aligned}$$

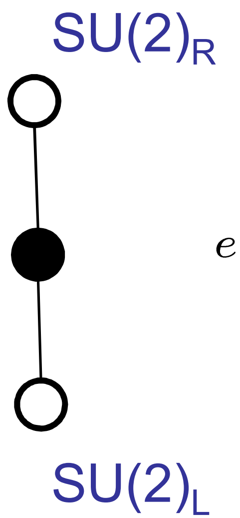
$$S_{PCF}(\theta_1 - \theta_2) = S_0^2(\theta_1 - \theta_2)\; S_{SU(2)}(\theta_1 - \theta_2) \times S_{SU(2)}(\theta_1 - \theta_2)$$

- Scalar (dressing) factor: $$S_0(\theta) = \frac{\Gamma\left(-\frac{\theta}{2i}\right)\Gamma\left(\frac{1}{2}+\frac{\theta}{2i}\right)}{\Gamma\left(\frac{\theta}{2i}\right)\Gamma\left(\frac{1}{2}-\frac{\theta}{2i}\right)}$$

Al&A.Zamolodchikov
Polyakov,Wiegmann
Faddeev,Reshetikhin

Asymptotic Bethe Ansatz (ABA) eqs., L $\to \infty$

- Bethe equations from periodicity $\quad e^{-iRm\,\sinh \pi\theta_k} = \prod_j{}' \widehat{S}(\theta_k - \theta_j)$



$$
\begin{aligned}
1 &= \prod_{\beta}^{J_u} \frac{u_j - \theta_\beta - i/2}{u_j - \theta_\beta + i/2} \prod_{i\neq j}^{J_u} \frac{u_j - u_i + i}{u_j - u_i - i}\,, \\
e^{-imR\sinh \pi\theta_\alpha} &= \prod_{\beta\neq\alpha}^{L} S_0^2\left(\theta_\alpha - \theta_\beta\right) \prod_{j}^{J_u} \frac{\theta_\alpha - u_j + i/2}{\theta_\alpha - u_j - i/2} \prod_{k}^{J_v} \frac{\theta_\alpha - v_k + i/2}{\theta_\alpha - v_k - i/2}\,, \\
1 &= \prod_{\beta}^{J_v} \frac{v_k - \theta_\beta - i/2}{v_k - \theta_\beta + i/2} \prod_{l\neq k}^{J_v} \frac{v_k - v_l + i}{v_k - v_l - i}\,,
\end{aligned}
$$

- θ-variables describe U(1)-sector (main circle of S^3 in O(4) model), u,v “magnon” variables – the transverse excitations on S^3, or SU(2)xSU(2)

$$
\begin{aligned}
E &= \sum_{k=1}^{N} m \cosh \pi\theta_k \\
P &= \sum_{k=1}^{N} m \sinh \pi\theta_k
\end{aligned}
$$

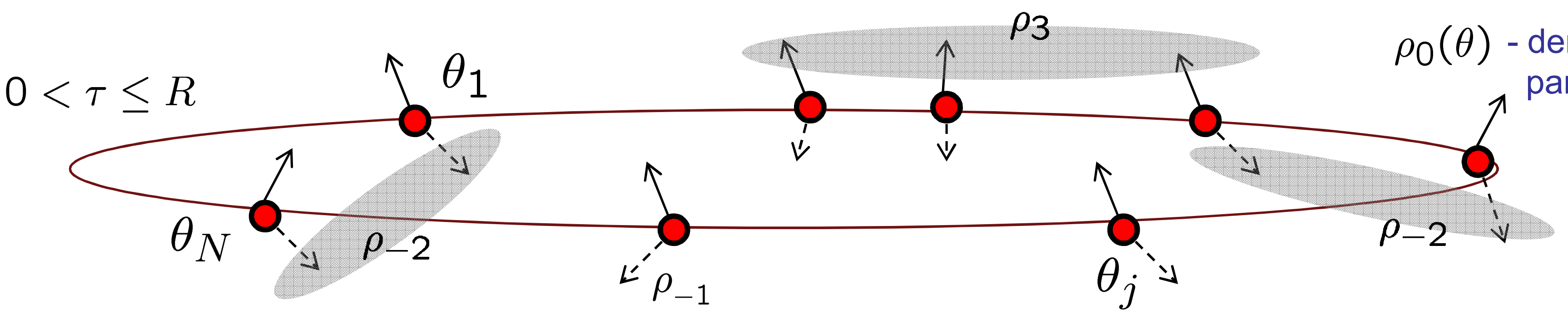
R →∞ : complex formation in infinite volume for SU(2) PCF

- Magnon bound states for $SU(2)_L$ and $SU(2)_R$
  in full analogy with Heisenberg chain

$$u_j^{(k)} = u^{(k)} + \frac{i}{2} j, \qquad j = -(k-1), -(k-3), \cdots, (k-1)$$

$\rho_n(\theta)$ - density of n-complex

$\rho_0(\theta)$ - density of particles



- Thermodynamic Bethe equations for densities (diagonalizing S-matrix of N particles)

$$\rho_n + \bar{\rho}_n = \delta_{n,0}\, p_0' + K_{nm} * \rho_m$$

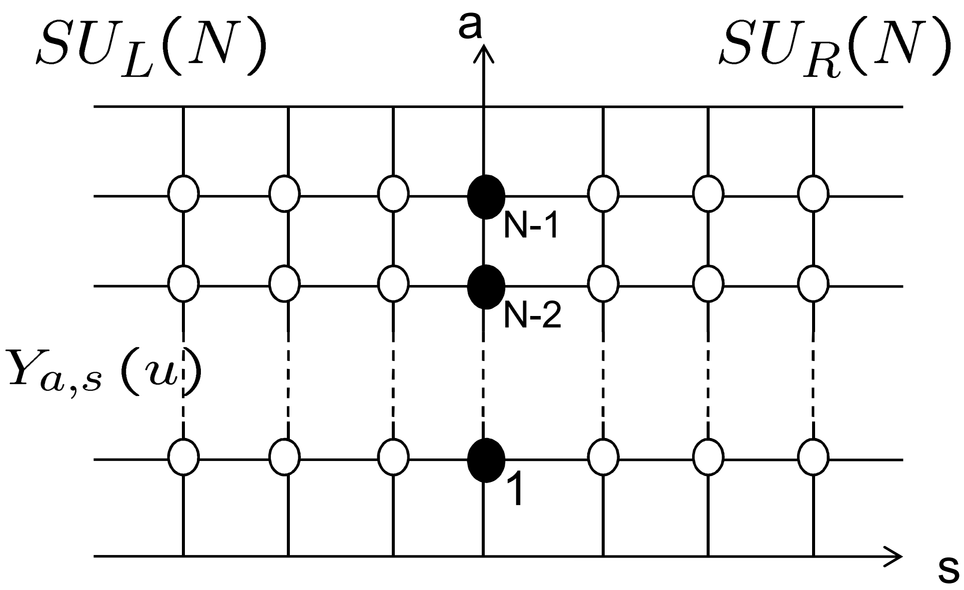
- TBA: Minimizing free energy at finite temperature T=1/L : TBA eqs. and Y-system

Y_n

$$f = \int \rho_0 \, E_0 \quad - \quad \frac{1}{L} \sum_{n=-\infty}^{\infty} \int \left[\rho_n \log\left(1 + \frac{\bar{\rho}_n}{\rho_n}\right) + \bar{\rho}_n \log\left(1 + \frac{\rho_n}{\bar{\rho}_n}\right) \right]_*$$

Y-system for SU(N)$_L$ x SU(N)$_R$ principal chiral field at finite L

Fateev,Onofri,Zamolodchikov'93



$$f^{\pm} = f(u \pm i/2)$$

$$\frac{Y^+_{a,s}Y^-_{a,s}}{Y_{a+1,s}Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

$$Y^a_s(u) \to e^{-L\epsilon_a(u)\delta_{s,0}} \times \mathrm{const}_{a,s}.$$

$$u \to \infty, \qquad \epsilon_a = m \sin\frac{\pi a}{N}\cosh\frac{2\pi u}{N}$$

- Energy (function of mL): $$E(mL) = -\frac{1}{N}\sum_{a=1}^{N-1}\int_{-\infty}^{\infty} du\, \epsilon_a(u)\log\left(1+Y^a_0(u)\right) + \sum_{j=1}^{\mathcal{N}}\epsilon_1(u_j)$$

- Finite size Bethe equation $1 + Y^1_0(u_j + iN/4) = 0$
- Relation to T-system (Hirota):

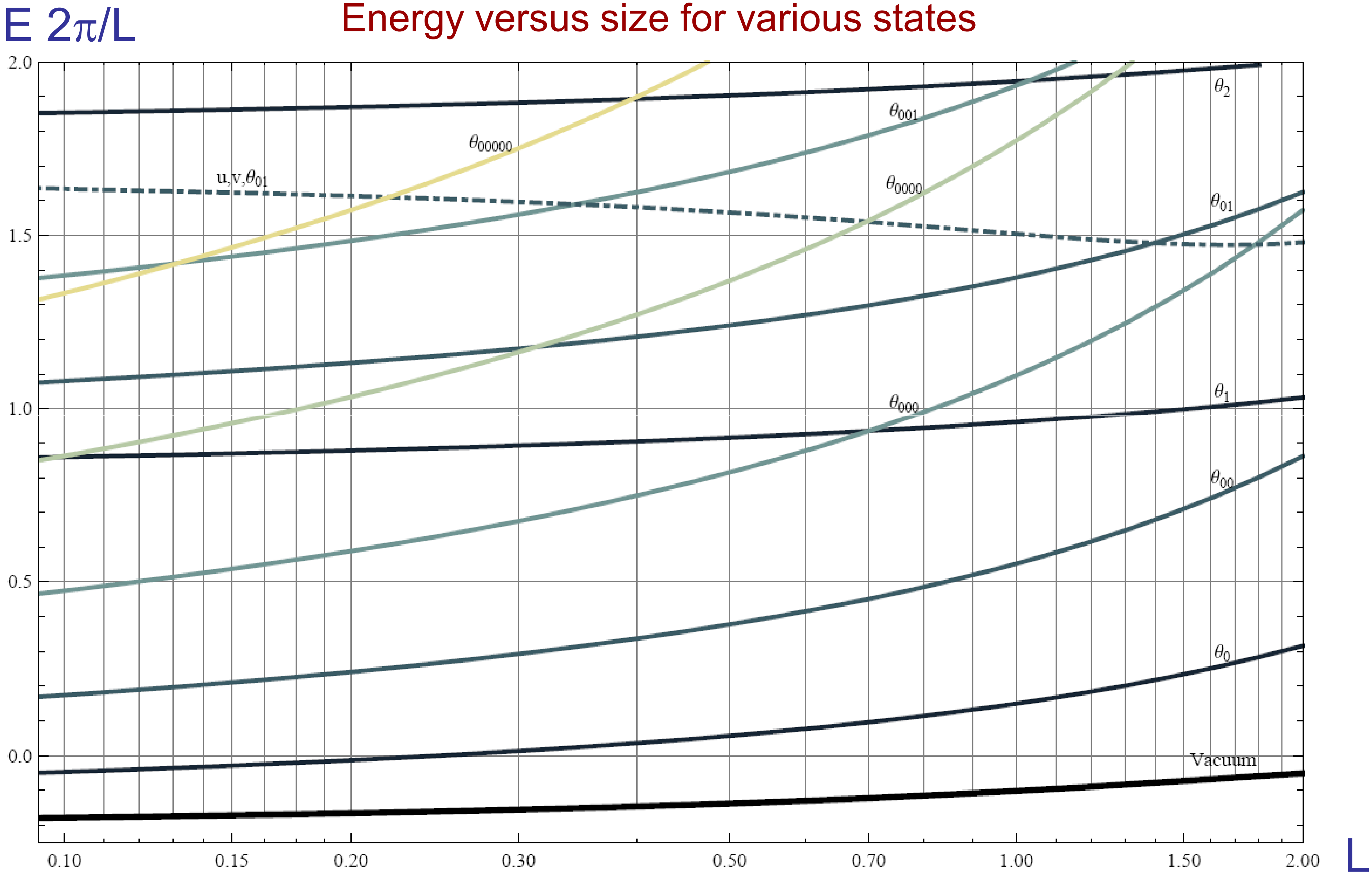
$$T^+_{a,s}T^-_{a,s} = T_{a,s+1}T_{a,s-1} + T_{a+1,s}T_{a-1,s}$$

$$Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}$$

- Discrete classical integrable dynamics!

Gromov,V.K.,Vieira'08

SU(2) PCF numerics (using our Destri-DeVega type eq.): Energy versus size for various states



• For vacuum and mass gap in accordance with **Balog,Hegedus'00'02**

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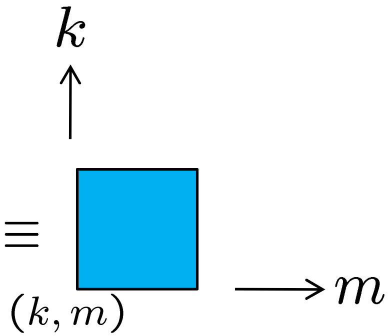
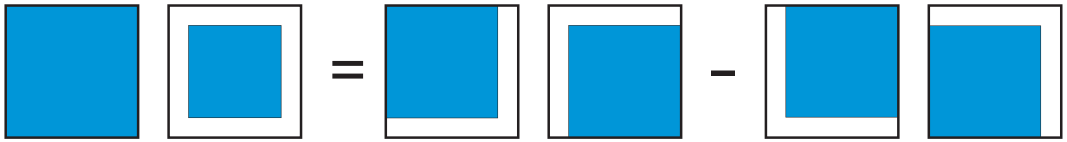
Hirota eq. from Jacobi relation

- Definition $\mathrm{det}_{\mathrm{N\times N}}(k,m) \equiv$



- Jacobi relation for determinants:



- By a linear map $N, k, m \longrightarrow u, a, s$

  we get integrable Hirota eq. (the Master Equation of Integrability!)

$$T_{a,s}(u+1)T_{a,s}(u-1) = T_{a,s-1}(u)\,T_{a,s+1}(u) + T_{a+1,s}(u)\,T_{a-1,s}(u)$$

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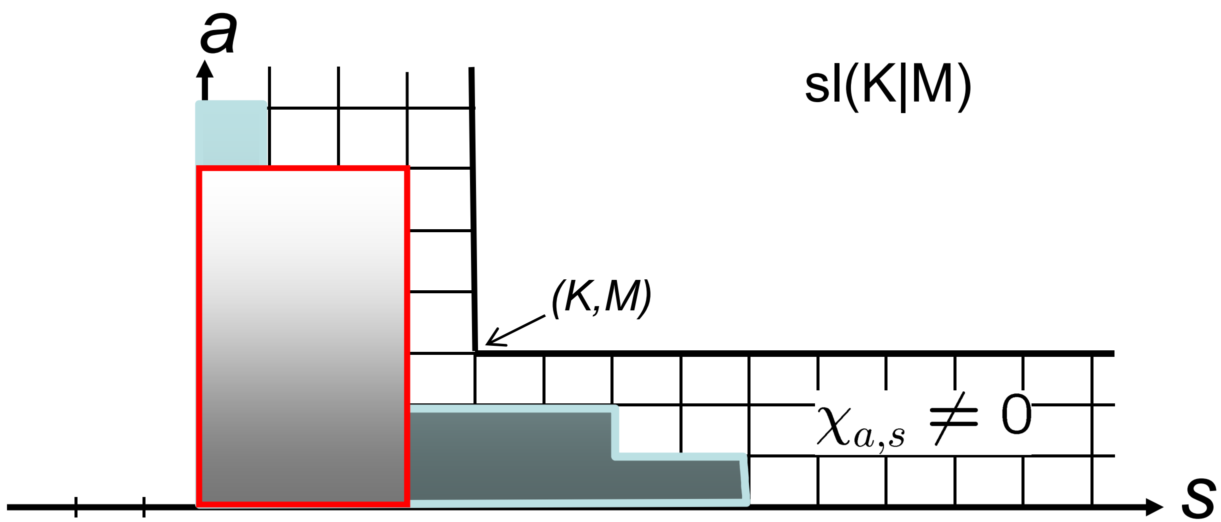
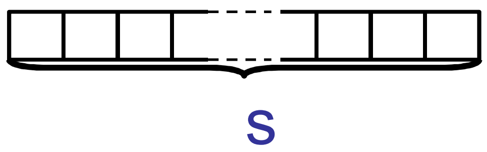
Fat Hook for Representations of sl(K|M)



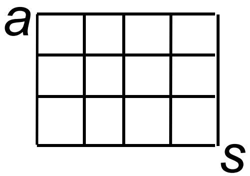
- All super Young tableaux of sl(K|M) live within this fat hook

- Jacobi-Trudi formula for SL(K|M) characters for general irrep $\lambda=\{\lambda_1,\lambda_2,\dots,\lambda_a\}$ in terms of characters for symmetric irreps: $\chi_s(g), \quad g \in sl(K|M)$

$$\chi_{\{\lambda\}}(g) = \det_{1\le i,j\le a} \chi_{\lambda_i - i + j}(g)\,.$$

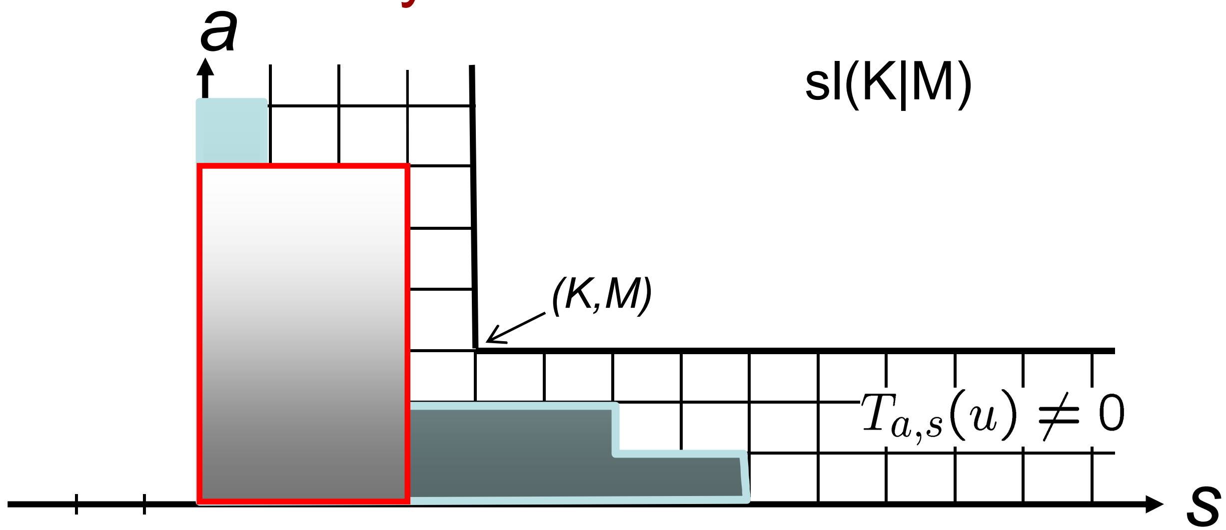


- For rectangular Young tableaux (a,s) Hirota eq. with fat hook b.c.:



$$\chi_{a,s}^2 = \chi_{a+1,s}\chi_{a-1,s} + \chi_{a,s+1}\chi_{a,s-1}$$

From Y-system to T-system (Hirota eq.)
SUSY Boundary Conditions: Fat Hook



Bazhanov,Reshetikhin'90
Cherednik'87
V.K.,Vieira'07 (proved,super, twisted)

- Determinant f-la for transfer matrices of rational sl(K|M) N-(super)spin chain

$$T^{\{\lambda\}}(u) = \frac{1}{S_N(u)} \det_{1\le k,j\le a} T_{\lambda_k-k+j}(u+i-ik)$$

- Hence the transfer matrices of rectangular irreps also satisfy Hirota eq.:

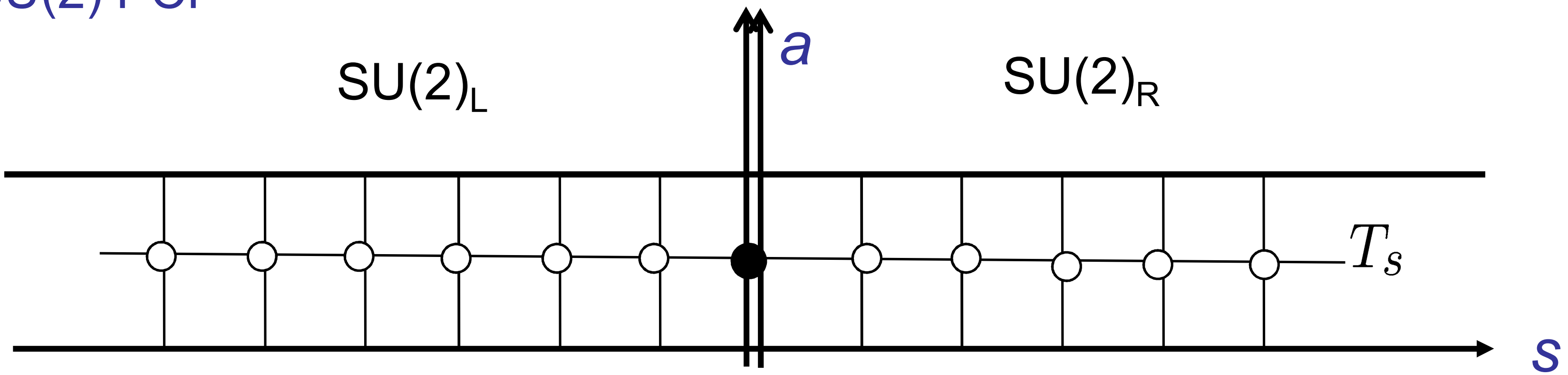
$$T_{a,s}(u+i/2)T_{a,s}(u-i/2) = T_{a,s-1}(u)\,T_{a,s+1}(u)+T_{a+1,s}(u)\,T_{a-1,s}(u)$$

- We can solve it by Bäcklund trick, find the full system of T-Q Baxter rel., as well as new Q-Q relations for quantum super-spin chains.

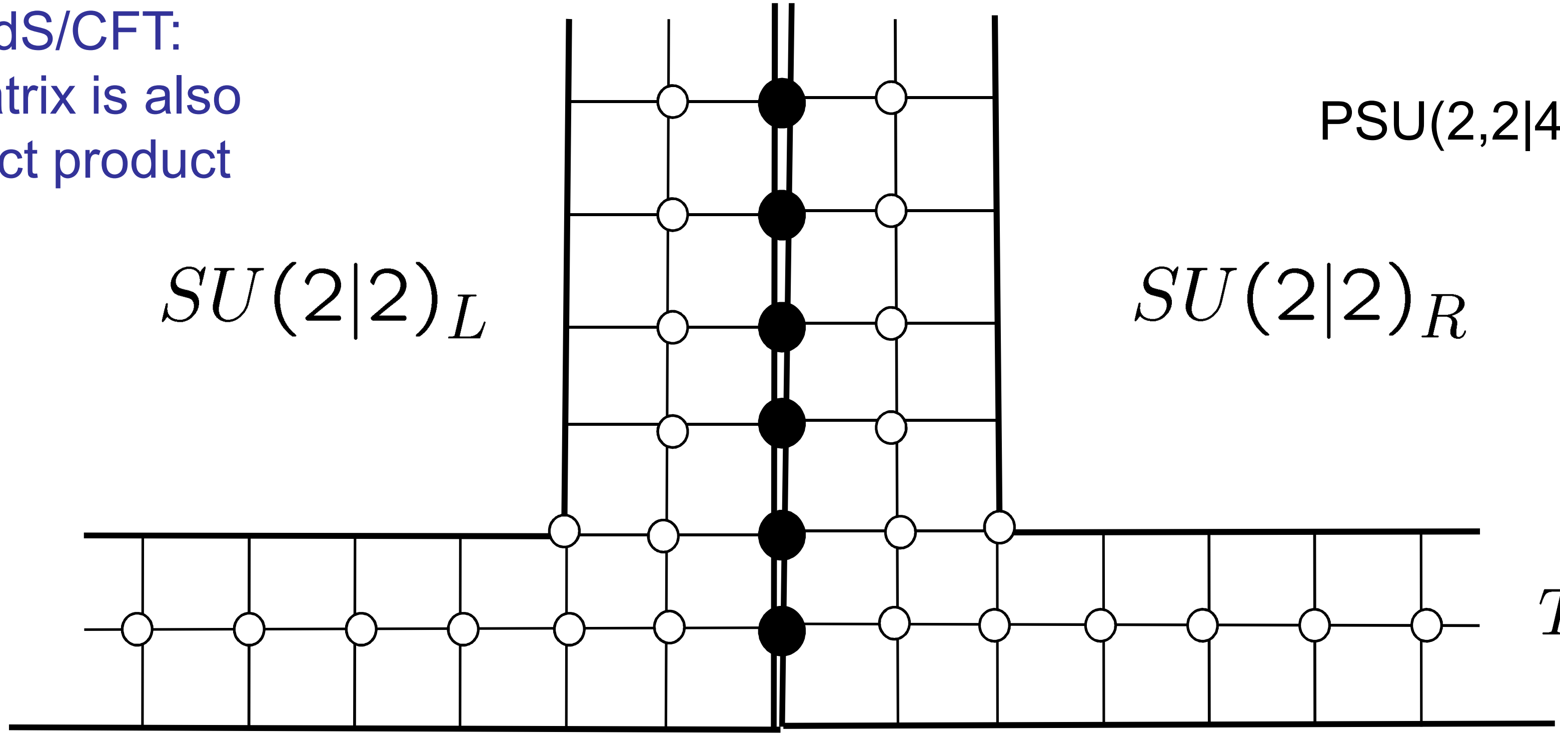
Krichever,Lupan,
Wiegmann,Zabrodin'96
Tsuboi'98,09
V.K.,Sorin,Zabrodin'07
Hegedus'09

Gluing T-hook out of two SU(2|2) fat hooks.
Integrability=Global Hirota eq., and presumably PSU(2,2|4)

For SU(2) PCF



For AdS/CFT:
S-matrix is also
a direct product



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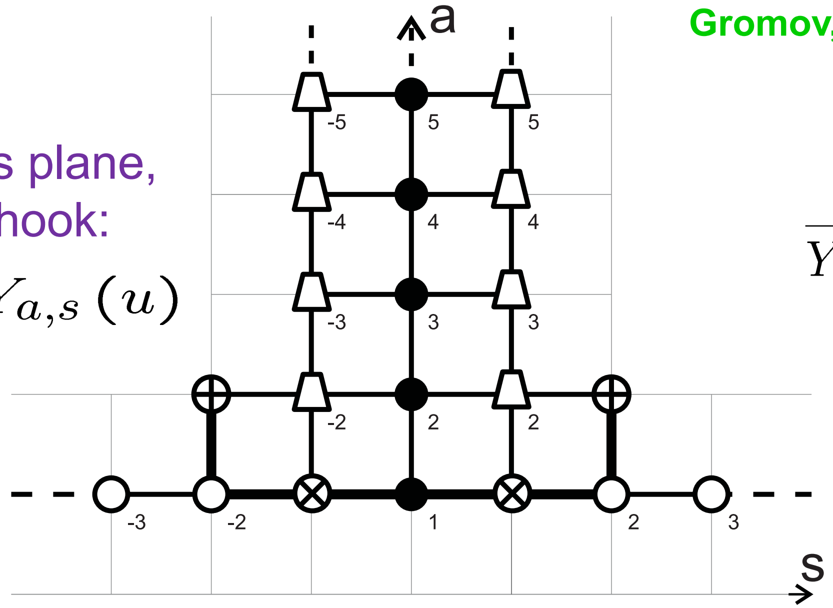
Y-system for full spectrum in AdS/CFT

Gromov,V.K.,Vieira'09

a,s plane,
T-hook:

$Y_{a,s}(u)$



$$f^{\pm} = f(u \pm i/2)$$

$$\frac{Y_{a,s}^{+} Y_{a,s}^{-}}{Y_{a+1,s} Y_{a-1,s}} = \frac{\left[1+Y_{a,s+1}\right]\left[1+Y_{a,s-1}\right]}{\left[1+Y_{a+1,s}\right]\left[1+Y_{a+1,s}\right]}$$

$$Y_{a,s\neq 0}(u\to\infty) \to \mathrm{const}_{a,s}$$

$$Y_{a,0}(u\to\infty) \to \left(\frac{x^{[-a]}}{x^{[+a]}}\right)^{L} \times \mathrm{const}_a \to 0$$

- Energy (dimension):

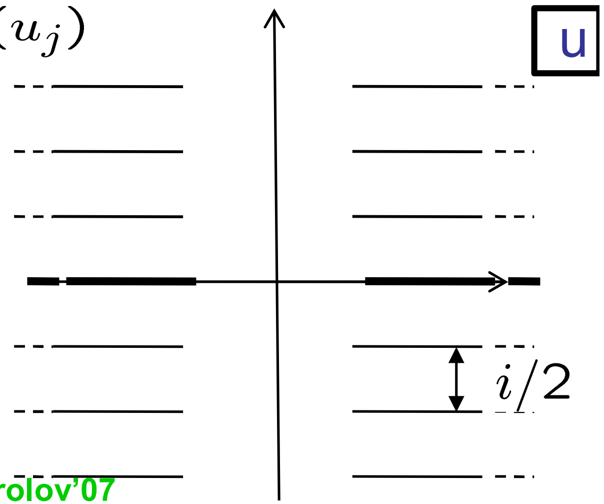
$$E_{state} - J = \int \frac{du}{2\pi i}\, \partial_u \epsilon_a^*(u) \log\left(1+Y_{a,0}^*(u)\right) + \sum_{j=1}^{M} \epsilon_1(u_j)$$

- Finite size Bethe eq.:

$$Y_{1,0}(u_j) + 1 = 0, \qquad j = 1, 2, \ldots, M$$

- 1-particle dispersion

$$\begin{cases} \epsilon_a(u) = \dfrac{2gi}{x^{[+a]}} - \dfrac{2gi}{x^{[-a]}} + a \\ p_a(u) = -i \log \dfrac{x^{[+a]}}{x^{[-a]}} \end{cases}$$



Arutyunov,Frolov'07

Gromov,V.K.,Vieira'09

Back to asymptotic Bethe eqs.: L⟶∞

$$Y_{a\geq 1,0} \sim \left(\frac{x^{[-a]}}{x^{[+a]}}\right)^L \to 0$$

$$1+Y_{a,s} = \frac{T^+_{a,s}T^-_{a,s}}{T_{a+1,s}T_{a-1,s}}$$

- It is a spin chain limit:

$$\frac{Y^+_{a,0}Y^-_{a,0}}{Y_{a-1,0}Y_{a+1,0}} \simeq \left(\frac{T^+_{a,1}T^-_{a,1}}{T_{a-1,1}T_{a+1,1}}\right)\left(\frac{T^+_{a,-1}T^-_{a,-1}}{T_{a-1,-1}T_{a+1,-1}}\right)$$

- T-system splits into two $SU(2|2)_{L,R}$ wings: $T_{a,s>0}$, $T_{a,s<0}$

- Solving this discrete Laplace eq. in (a,u)-variables we get

$$Y_{a,0}(u) \simeq \left(\frac{x^{[-a]}}{x^{[+a]}}\right)^L \frac{\phi^{[-a]}}{\phi^{[+a]}}\, T^L_{a,-1}T^R_{a,1}$$

transfer matrices of SU(2|2)

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Asymptotic Bethe Ansatz of Beisert-Staudacher $L \to \infty$

- Fundamental transfer matrix for $SU(2|2)_{L,R}$

$$T_{1,1} = \frac{R^{-(+)}}{R^{-(-)}}\left[-\frac{R^{-(-)}Q_3^+}{R^{-(+)}Q_3^-} + \frac{Q_2^{--}Q_3^+}{Q_2 Q_3^-} + \frac{Q_2^{++}Q_1^-}{Q_2 Q_1^+} - \frac{Q_1^- B^{+(+)}}{Q_1^+ B^{+(-)}}\right]$$

Beisert'06,

$$\overset{3}{\otimes} - - - \overset{2}{\bigcirc} - - - \overset{1}{\otimes}$$

Definitions of Baxter functions:

$$R_l^{(\pm)}(u) \equiv \prod_{j=1}^{K_l} \frac{x(u) - x_{l,j}^{\mp}}{(x_{l,j}^{\mp})^{1/2}}$$

$$B_l^{(\pm)}(u) \equiv \prod_{j=1}^{K_l} \frac{\frac{1}{x(u)} - x_{l,j}^{\mp}}{(x_{l,j}^{\mp})^{1/2}}$$

$$Q_l(u) = \prod_{j=1}^{J_l}(u - u_{l,j}) = -R_l(u)B_l(u)$$

$$l = 1,2,3(5,6,7)$$

- Ansatz, to fit ABA of Beisert-Staudacher eq.

$$Y_{1,0}(u_{4,j}) = -1$$

$$\frac{\phi^-}{\phi^+} = S^2\,\frac{B^{+(+)}R^{-(-)}}{B^{-(-)}R^{+(+)}}\,\frac{B_7^+ B_5^-}{B_7^- B_5^+}\,\frac{B_1^+ B_3^-}{B_1^- B_3^+}$$

- Fusion (solving Hirota!) generates the rest of $T_{a,s\neq 0}(u)$
- Crossing $\leftrightarrow$ invariance of Y's under particle-antiparticle transf. $x^{\pm} \to 1/x^{\pm}$

$$\sigma_{12}\sigma_{\bar{1}2} = \frac{x_2^-}{x_2^+}\,\frac{x_1^- - x_2^-}{x_1^+ - x_2^-}\,\frac{1/x_1^- - x_2^+}{1/x_1^+ - x_2^+}$$

Janik'06

- $S(u) = \prod_j \sigma(x(u), x_{4,j})$ derived from crossing Volin'09

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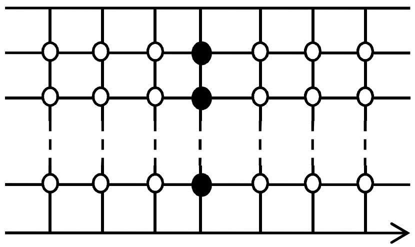
From Y/T-system to Destri-deVega-like eq. for AdS/CFT

- We can profit of the classical integrability of Y/T-system, as we did it for $SU(2)_L \times SU(2)_R$ PCF. Determinant solution in a strip:

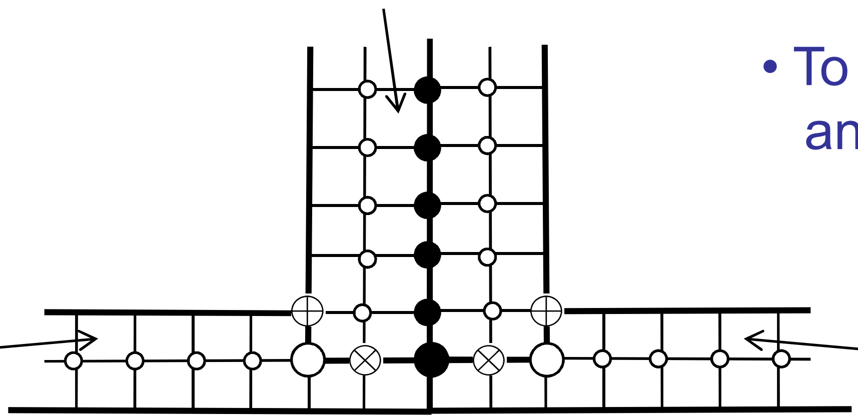


Krichever,Lupan, Wiegmann,Zabrodin'96

$$T_k(x) = h(x+ik/2)\begin{vmatrix} Q(x+i\frac{k+1}{2}) & R(x+i\frac{k+1}{2}) \\ \bar{Q}(x-i\frac{k+1}{2}) & \bar{R}(x-i\frac{k+1}{2}) \end{vmatrix}$$

- Analyticity and L↔R symmetry fix a DdV-like equation. Gromov,V.K.,Vieira'08
- For SU(N) PCF: N×N det solution and analyticity also available. Leurent,V.K.,Vicedo Work in progress
- Same program should be possible in AdS/CFT:

SU(4) det



SU(2) det

SU(2) det

- To be supplemented by analyticity and L↔R symmetry arguments

All three glued together by Hirota along Kac-Dynkin nodes

Gromov,Leurent,V.K.,Vieira, Work in progress

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Comments

- Y-system for all operators in AdS/CFT is constructed in functional and integral form
- Tested in weak coupling: we reproduced the 4-loop results.
- Not solved yet in strong coupling, but our numerical solution of Y-system for Konishi confirms the supergravity predictions and predicts next coefficients of SC expansion. (Kolya's talk)
- Did we miss mu-term? No signs of it so far...

To do:

- Destri-deVega-type equation would be the best numerical tool (in the spirit of **Gromov,V.K.,Vieira'08** for SU(2) PCF sigma-model)
- Derive integral Y-system from functional (in the opposite way it works)
- What are Y's on the SYM side? Can we derive Y-system from SYM?
- BFKL
- 3D Super-Chern-Simons integral Y-system
- Lessons for QCD ?

END