Yangian Symmetry of Scattering Amplitudes in $\mathcal{N} = 4$ Super Yang-Mills

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The setting

AdS/CFT correspondence: Fascinating link between conformal quantum field theories without gravity and string theory with (both classical and quantized) gravity

Two major (recent) developments in the maximal susy $\mathrm{AdS}_5/\mathrm{CFT}_4$ system:

1. Integrability in AdS/CFT:
   $\Rightarrow$ solution of the spectral problem
2. Scattering amplitudes in maximally susy Yang-Mills,
   $\Rightarrow$ relation to light-like Wilson loops
   $\Rightarrow$ emergence of dual superconformal symmetry

This talk: Can we connect the two?

$\mathcal{N}=4$ super Yang Mills: The simplest interacting 4d QFT

- Field content: All fields in adjoint of $SU(N)$, $N \times N$ matrices
  - Gluons: A_μ, $\mu = 0,1,2,3$, $\Delta = 1$
  - 6 real scalars: Φ_I, $I = 1,\ldots,6$, $\Delta = 1$
  - 4×4 real fermions: $\Psi_{\alpha\,A}$, $\bar{\Psi}^{\dot{\alpha}}_{A}$,$\alpha, \dot{\alpha} = 1,2$. $A = 1,2,3,4$, $\Delta = 3/2$
  - Covariant derivative: $\mathcal{D}_\mu = \partial_\mu - i[A_\mu, *]$, $\Delta = 1$

- Action: Unique model completely fixed by SUSY

$$S = \frac{1}{{g_{\mathrm{YM}}}^2} \int d^4x \,\mathrm{Tr}\Big[\tfrac{1}{4}F_{\mu\nu}^2 + \tfrac{1}{2}(D_\mu\Phi_i)^2 - \tfrac{1}{4}[\Phi_I,\Phi_J][\Phi_I,\Phi_J] + \bar{\Psi}^A_{\dot{\alpha}}\sigma^{\dot{\alpha}\beta}_\mu \mathcal{D}^\mu \Psi_{\beta\,A} - \tfrac{i}{2}\Psi_{\alpha\,A}\sigma^{AB}_I\epsilon^{\alpha\beta}\,[\Phi^I, \Psi_{\beta\,B}] - \tfrac{i}{2}\bar{\Psi}_{\dot{\alpha}\,A}\sigma^{AB}_I\epsilon^{\dot{\alpha}\dot{\beta}}\,[\Phi^I, \bar{\Psi}_{\dot{\beta}\,B}]\Big]$$

- $\boxed{\beta_{g_{\mathrm{YM}}} = 0}$: Quantum Conformal Field Theory, 2 parameters: N & $\lambda = {g_{\mathrm{YM}}}^2 N$

- Shall consider 't Hooft planar limit: $N \to \infty$ with λ fixed.

Most symmetric 4d gauge theory!

- Symmetry: $\mathfrak{so}(2,4) \otimes \mathfrak{so}(6) \subset \mathfrak{psu}(2,2|4)$

Poincaré: $p^{\alpha\dot{\alpha}} = p_\mu\,(\sigma^\mu)^{\dot{\alpha}\beta}, \quad m_{\alpha\beta}, \quad \bar{m}_{\dot{\alpha}\dot{\beta}}$

Conformal: $k_{\alpha\dot{\alpha}}, \quad d \qquad (c$: central charge$)$

R-symmetry: r_{AB}

Poncaré Susy: $q^{\alpha\,A}, \bar{q}^{\dot{\alpha}}_A$ Conformal Susy: $s_{\alpha\,A}, \bar{s}^A_{\dot{\alpha}}$

- $4+4$ Supermatrix notation $\bar{A} = (\alpha, \dot{\alpha}|A)$

$$J^{\bar{A}}{}_{\bar{B}} = \begin{pmatrix} m^\alpha{}_\beta - \frac{1}{2}\,\delta^\alpha_\beta\,(d + \frac{1}{2}c) & k^\alpha{}_{\dot{\beta}} & s^\alpha{}_B \\ p^{\dot{\alpha}}{}_\beta & \overline{m}^{\dot{\alpha}}{}_{\dot{\beta}} + \frac{1}{2}\,\delta^{\dot{\alpha}}_{\dot{\beta}}\,(d - \frac{1}{2}c) & \bar{q}^{\dot{\alpha}}{}_B \\ q^A{}_\beta & \bar{s}^A{}_{\dot{\beta}} & -r^A{}_B - \frac{1}{4}\delta^A_B\,c \end{pmatrix}$$

- Algebra:

$$[J^{\bar{A}}{}_{\bar{B}}\,,\, J^{\bar{C}}{}_{\bar{D}}\} = [\,\delta^{\bar{C}}_{\bar{B}}\,J^{\bar{A}}{}_{\bar{D}} - (-1)^{(|\bar{A}|+|\bar{B}|)(|\bar{C}|+|\bar{D}|)}\delta^{\bar{A}}_{\bar{D}}\,J^{\bar{C}}{}_{\bar{B}}]$$

Observables

- **Local operators:** $\mathcal{O}_n(x) = \mathrm{Tr}[\mathcal{W}_1\,\mathcal{W}_2\ldots\mathcal{W}_n]$ with $\mathcal{W}_i \in \{\mathcal{D}^k\Phi, \mathcal{D}^k\Psi, \mathcal{D}^k F\}$

  2 point fct: $\langle \mathcal{O}_a(x_1)\,\mathcal{O}_b(x_2)\rangle = \frac{\delta_{ab}}{(x_1-x_2)^{2\,\Delta_a(\lambda)}}$ $\quad \Delta_a(\lambda)$ Scaling Dims

  3 point fct: $\Big\langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\mathcal{O}_c(x_2)\Big\rangle = \frac{c_{abc}(\lambda)}{x_{12}^{\Delta_a+\Delta_b-\Delta_c}\,x_{23}^{\Delta_b+\Delta_c-\Delta_a}\,x_{31}^{\Delta_c+\Delta_a-\Delta_b}}$

  n-point functions follow from OPE

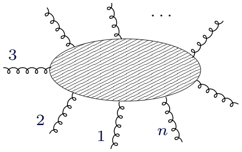
- **Wilson loops:**

$$\mathcal{W}_C = \Big\langle \mathrm{Tr} P \exp i \oint_C ds\,(\dot{x}^\mu A_\mu + i|\dot{x}|\,\theta^I\,\Phi_I) \Big\rangle$$

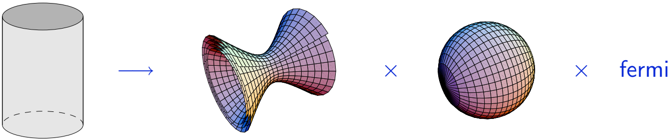
- **Scattering amplitudes:**

$$\mathcal{A}_n(\{p_i, h_i, a_i\};\lambda) = \begin{Bmatrix} \text{UV-finite} \\ \text{IR-divergent} \end{Bmatrix}$$

  helicities: $h_i \in \{0, \pm\frac{1}{2}, \pm 1\}$



Superstring in $AdS_5 \times S^5$



$$I = \sqrt{\lambda} \int d\tau\, d\sigma \left[G^{(\mathrm{AdS}_5)}_{mn}\, \partial_a X^m \partial^a X^n + G^{(\mathrm{S}_5)}_{mn}\, \partial_a Y^m \partial^a Y^n + \text{fermions} \right]$$

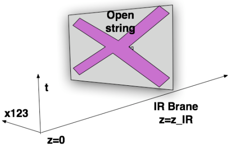
- $ds^2_{AdS} = R^2 \dfrac{dx^2_{3+1} + dz^2}{z^2}$ has boundary at $z = 0$
- $\sqrt{\lambda} = \frac{R^2}{\alpha'}$, classical limit: $\sqrt{\lambda} \to \infty$, quantum fluctuations: $\mathcal{O}(1/\sqrt{\lambda})$
- $AdS_5 \times S^5$ is max susy background (like $\mathbb{R}^{1,9}$ and plane wave)
- **Quantization unsolved!**
- String coupling constant $g_s = \frac{\lambda}{4\pi N} \to 0$ in 't Hooft limit
- Isometries: $\mathfrak{so}(2,4) \times \mathfrak{so}(6) \subset \mathfrak{psu}(2,2|4)$
- Include fermions: Formulate as $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ supercoset model [Metsaev,Tseytlin]

Gauge Theory - String Theory Dictionary of Observables

$\Delta_a(\lambda)$ spectrum of scaling dimensions $\Leftrightarrow$ $E(\lambda)$ string excitation spectrum solved (?)

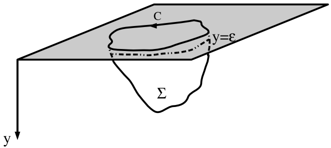
$c_{abc}(\lambda)$ structure constants $(\Leftrightarrow)$ Only SUGRA: $\mathcal{Z}_{AdS}[\phi|_{\partial \text{AdS}} = J] = \mathcal{Z}_{\text{CFT}}[J]$

$\mathcal{A}_n(\{p_i, h_i, a_i\}; \lambda)$ $(\Leftrightarrow)$



open string amplitude

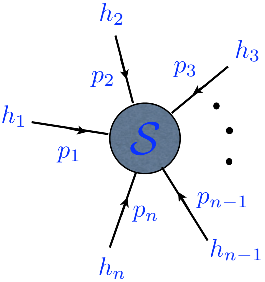
Wilson loop $\mathcal{W}_C$ $\Leftrightarrow$



minimal surface

Scattering amplitudes in $\mathcal{N} = 4$ SYM I

- Consider n-particle scattering amplitude



Planar amplitudes most conveniently expressed in color ordered formalism:

$$A_n(\{p_i, h_i, a_i\}) = (2\pi)^4 \, \delta^{(4)}(\sum_{i=1}^{n} p_i) \sum_{\sigma \in S_n/Z_n} g^{n-2} \, \mathrm{tr}[t^{a_1} \ldots t^{a_n}]$$
$$\times \mathcal{A}_n(\{p_{\sigma_1}, h_{\sigma_1}\}, \ldots, \{p_{\sigma_1}, h_{\sigma_1}\}; \lambda = g^2 \, N)$$

$\mathcal{A}_n$: Color ordered amplitude. Color structure is stripped off.

Helicity of ith particle: $h_i = 0$ scalar, $h_i = \pm 1$ gluon, $h_i = \pm\frac{1}{2}$ gluino

Spinor helicity formalism

- Express momentum and polarizations via commuting spinors $\lambda^\alpha,\ \tilde{\lambda}^{\dot\alpha}$:

$$p^{\alpha\dot\alpha} = (\sigma^\mu)^{\alpha\dot\alpha}\, p_\mu = \lambda^\alpha \tilde{\lambda}^{\dot\alpha} \quad \Leftrightarrow \quad p_\mu\, p^\mu = \det p^{\alpha\dot\alpha} = 0$$

- Choice of helicity determines polarization vector ε^μ of gluon

$$h = +1 \qquad \varepsilon^{\alpha\dot\alpha} = \frac{\lambda^\alpha \tilde{\mu}^{\dot\alpha}}{[\tilde{\lambda}\,\tilde{\mu}]} \qquad [\tilde{\lambda}\,\tilde{\mu}] := \epsilon_{\dot\alpha\dot\beta}\, \tilde{\lambda}^{\dot\alpha} \tilde{\mu}^{\dot\beta}$$

$$h = -1 \qquad \tilde{\varepsilon}^{\alpha\dot\alpha} = \frac{\mu^{\alpha} \tilde{\tilde{\alpha}}^{\dot\alpha}}{\langle\lambda\,\mu\rangle} \qquad \langle\lambda\,\mu\rangle := \epsilon_{\alpha\beta}\, \lambda^\alpha \mu^\beta$$

$\mu, \bar{\mu}$ arbitrary reference spinors.

- E.g. scalar products: $p_1 \cdot p_2 = \langle\lambda_1, \lambda_2\rangle\, [\tilde{\lambda}_2, \tilde{\lambda}_1] = \langle 1, 2\rangle\, [2, 1]$

Scattering amplitudes in $\mathcal{N} = 4$ SYM II

- Gluon amplitudes: $\mathcal{A}_n(1^+, 2^+, \ldots, n^+) = 0 = \mathcal{A}_n(1^-, 2^+, \ldots, n^+)$ by SUSY Ward identity
- Maximally helicity violating (MHV) amplitudes

$$\mathcal{A}_n(1^- 2^+, \ldots, (j-1)^+, j^-, (j+1)^+, \ldots n^+) = \mathcal{A}^{\text{MHV}}_{n;0} + \lambda \cdot \mathcal{A}^{\text{MHV}}_{n;1} + \lambda^2 \cdot \mathcal{A}^{\text{MHV}}_{n;2} + \ldots = \mathcal{A}^{\text{MHV}}_{n;0} \cdot \mathcal{M}^{\text{MHV}}_n(\{p_i \cdot p_j\}; \lambda)$$

Parke-Taylor formula:

$$\mathcal{A}^{\text{MHV}}_{n;0} = i \frac{\langle 1, j \rangle^4}{\langle 1, 2 \rangle \langle 2, 3 \rangle \ldots \langle n, 1 \rangle}$$

[Parke,Taylor]

- BDS conjecture [Anastasiou,Bern,Dixon,Kosower; Bern,Dixon,Smirnov]

$$\log \mathcal{M}^{\text{MHV}}_n = \Gamma_{\text{cusp}}(\lambda) \cdot \mathcal{M}^{\text{MHV}}_{n,\text{1-loop,finite}} + \text{“}\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\text{”}$$

True for $n = 4, 5$ known to receive corrections for $n \geq 6$

[Drummond,Henn,Korchemsky,Sokatchev; Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich]

$\Rightarrow$ We know **all** 4 and 5 point amplitudes to all loop order!

- N^{k}MHV amplitudes have rather complicated structure! $\Rightarrow$ Better formulation?

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[Parke, Taylor]

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On-shell superspace

- Introduce Grassmann variables $\eta_i^A \quad A=1,2,3,4 \quad i=1,\ldots,n$ [Nair]
- Superwavefunction:

$$\Phi(p,\eta) = G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar{\Gamma}^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D \epsilon_{ABCD} G^-(p)$$

- Express amplitudes compactly in on-shell superspace $(\lambda_i^\alpha, \tilde{\lambda}^{\dot{\alpha}}, \eta_i^A)$
- MHV-superamplitude: Packaged gluon$^\pm$-gluino$^{\pm 1/2}$-scalar amplitude

$$\mathbb{A}_{n;0}^{\mathsf{MHV}}(\lambda_1,\tilde{\lambda}_1,\eta_1;\ldots;\lambda_n,\tilde{\lambda}_n,\eta_n) = i(2\pi)^4 \frac{\delta^{(4)}(\sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}})\,\delta^{(8)}(\sum_i \lambda_i^\alpha \eta_i^A)}{\langle 1,2\rangle\langle 2,3\rangle\ldots\langle n,1\rangle}$$

Conservation of 'fermionic' momentum: $\delta^{(8)}(\sum_i \lambda^\alpha \eta_i^A) = (\sum_i \lambda^\alpha \eta_i^A)^8$

η-expansion associates $(\eta_i)^m := \prod_{k=1}^m \eta_i^{A_k}$ with ith particle of helicity $1-m/2$

$$\Rightarrow \quad \mathbb{A}_n^{\mathsf{MHV}} = i(2\pi)^4\,\delta^{(4)}(\sum_i p_i)\sum_{j\neq k}(\eta_j)^4(\eta_k)^4\,\mathcal{A}_n^{\mathsf{MHV}}(1^+\ldots j^-\ldots k^-\ldots n^+)$$

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Superamplitudes and $\mathfrak{su}(2,2|4)$ invariance

- General form of superamplitudes:

$$\boxed{\mathbb{A}_n = i(2\pi)^4 \frac{\delta^{(4)}(\sum_i \lambda_i \tilde{\lambda}_i)\, \delta^{(8)}(\sum_i \lambda_i \eta_i)}{\langle 1,2\rangle \langle 2,3\rangle \ldots \langle n,1\rangle} \mathcal{P}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})}$$

- $\mathbb{A}_n$ invariant under superconformal group $\mathfrak{psu}(2,2|4)$:
$p, m, \bar{m}, k, d \oplus r \oplus q, \bar{q}, s, \bar{s} \oplus (c)$

$$\boxed{p^{\alpha\dot{\alpha}}\, \mathbb{A}_n = q^{\alpha A}\, \mathbb{A}_n = k_{\alpha\dot{\alpha}}\, \mathbb{A}_n = s_{\alpha A} \mathbb{A}_n = d\, \mathbb{A}_n = \ldots = 0}$$

- Realization of $\mathfrak{psu}(2,2|4)$ generators in on-shell superspace, e.g. [Witten]

$$p^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \lambda_i^\alpha\, \tilde{\lambda}_i^{\dot{\alpha}} \qquad q^{\alpha A} = \sum_{i=1}^{n} \lambda_i^\alpha\, \eta_i^A \qquad \Rightarrow \text{ obvious symmetries}$$

$$k_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i^\alpha} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \qquad s_{\alpha A} = \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i^\alpha} \frac{\partial}{\partial \eta_i^A} \qquad \Rightarrow \text{ less obvious sym}$$

- Also: **Local** invariance $\quad h_i\, \mathbb{A}_n = 1 \cdot \mathbb{A}_n$

Helicity operator: $\quad h_i = -\frac{1}{2}\, \lambda_i^\alpha\, \partial_{i\,\alpha} + \frac{1}{2}\, \tilde{\lambda}_i^{\dot{\alpha}}\, \partial_{i\,\dot{\alpha}} + \frac{1}{2}\, \eta_i^A\, \partial_{i\,A} = 1 - c_i$

$\Rightarrow \boxed{c_i\, \mathbb{A}_n = 0}$ (Tree) superamplitudes are $\mathfrak{su}(2,2|4)$ invariant

$\mathfrak{su}(2,2|4)$ invariance

- The $\mathfrak{su}(2,2|4)$ generators acting in on-shell superspace $(\lambda_i^\alpha, \tilde\lambda_i^{\dot\alpha}, \eta_i^A)$:

$$p^{\dot\alpha\alpha} = \sum_i \tilde\lambda_i^{\dot\alpha}\lambda_i^\alpha\,, \qquad k_{\alpha\dot\alpha} = \sum_i \partial_{i\alpha}\partial_{i\dot\alpha}\,,$$

$$\overline{m}_{\dot\alpha\dot\beta} = \sum_i \tilde\lambda_{i(\dot\alpha}\partial_{i\dot\beta)}, \qquad m_{\alpha\beta} = \sum_i \lambda_{i(\alpha}\partial_{i\beta)}\,,$$

$$d = \sum_i [\tfrac{1}{2}\lambda_i^\alpha\partial_{i\alpha} + \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} + 1], \qquad r^A{}_B = \sum_i [-\eta_i^A\partial_{iB} + \tfrac{1}{4}\delta_B^A\eta_i^C\partial_{iC}]\,,$$

$$q^{\alpha A} = \sum_i \lambda_i^\alpha\eta_i^A\,, \qquad \bar q_A^{\dot\alpha} = \sum_i \tilde\lambda_i^{\dot\alpha}\partial_{iA}\,,$$

$$s_{\alpha A} = \sum_i \partial_{i\alpha}\partial_{iA}, \qquad \bar s_{\dot\alpha}^A = \sum_i \eta_i^A\partial_{i\dot\alpha}\,,$$

$$c = \sum_i [1 + \tfrac{1}{2}\lambda_i^\alpha\partial_{i\alpha} - \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} - \tfrac{1}{2}\eta_i^A\partial_{iA}]\,.$$

- Invariance: $\{\,p, k, \bar m, m, d, r, q, \bar q, s, \bar s, c_i\,\}\,\mathbb{A}_n^{\text{tree}}(\lambda_i^\alpha, \tilde\lambda_i^{\dot\alpha}, \eta_i^A) = 0$
- N.B: Subtleties for colinear momenta due to holomorphic anomalies

[Bargheer, Beisert, Galleas, Loebbert,McLoughlin][Korchemsky, Sokatchev] [Skinner,Mason][Arkani-Hamed, Cachazo, Kaplan]

On shell recursion techniques

- Efficient way of computing tree level gluon amplitudes: BCFW On shell recursion techniques [Britto,Cachazo,Feng+Witten '04,05]

  Closed formula for 'split helicity' gluon amplitudes $(+\ldots+-\ldots-)$

  [Roiban,Spradlin,Volovich,Britto,Feng]

- Reformulation of recursion relations in on-shell superspace through shift in $(\lambda_i, \tilde{\lambda})$ and η_i [Elvang et al 08, Arkani-Hamed et al 08, Brandhuber et al 08]

- Recursion much simpler and can be solved!

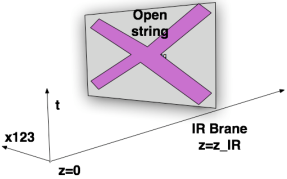
  [Drummond, Henn, Korchemsky, Sokatchev 08; Drummond,Henn 08]

$$\mathbb{A}_n = i(2\pi)^4 \frac{\delta^{(4)}(\sum_i \lambda_i \tilde{\lambda}_i)\, \delta^{(8)}(\sum_i \lambda_i\, \eta_i)}{\langle 1,2\rangle \langle 2,3\rangle \ldots \langle n,1\rangle}\, \mathcal{P}_n^{\text{tree}}(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})$$

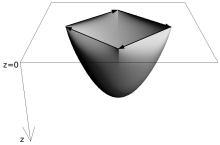
$\Rightarrow$ $\mathcal{P}_n^{\text{tree}}$ now known analytically (implies in particular pure Yang-Mills result!).

MHV Scattering amplitudes in AdS/CFT

- Dual string description of scattering amplitudes [Alday,Maldacena '07]



T-dual $\Leftrightarrow$



Open string amplitude on IR-brane $\overset{\text{T-dual}}{\Leftrightarrow}$ Wilson loop with light-like segments

- Cusp points determined by gluon momenta via key relation

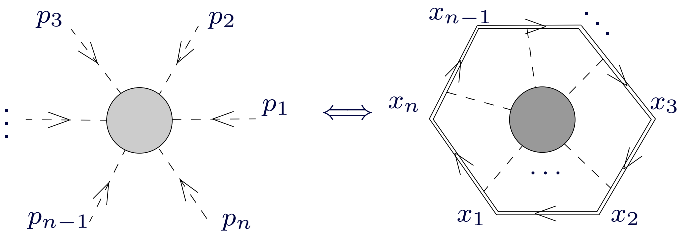
$$\boxed{p_i^\mu = x_{i+1}^\mu - x_i^\mu}$$

Is also seen in conformal 4-pt integrals [Drummond,Henn,Smirnov,Sokatchev 06]

- Yields strong coupling prediction for four-gluon MHV amplitude via classical string theory!
- Indeed BDS conjecture for $n = 4$ gluons tested:

$$\lim_{\lambda\to\infty} \log \mathcal{M}_4^{\text{MHV}} = \underbrace{\sqrt{\lambda}/2\pi}_{\Gamma_{\text{cusp}}(\lambda\to\infty)} \cdot \mathcal{M}_{n,\text{1-loop}}^{\text{MHV}} + \text{``}\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\text{''}$$

Scattering amplitude ⇔ Wilson loop duality at perturbative level



$$x^\mu_{i+1} - x^\mu_i = p^\mu_i$$

[Drummond,Henn,Korchemsky, Sokatchev]

Planar relation:

$$\ln \mathcal{M}_n^{\text{MHV}} = \ln \mathcal{W}_n + \text{div} + \mathcal{O}(\epsilon)$$

$$\mathcal{W}_n = \frac{1}{N} \left\langle \mathrm{Tr} P \exp[ig \oint_{C_n} dx^\mu \, A_\mu] \right\rangle$$

Checked up to two loops and $n \leq 6$ points

[Drummond,Henn,Korchemsky,Sokatchev;Brandhuber,Heslop,Travaglini; Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich]

String interpretation: Combination of bosonic and ‘fermionic’ T-duality transformation for $AdS_5 \times S^5$ superstring.

[Beisert,Ricci,Tseytlin,Wolf;Berkovits,Maldacena]

⇒ Conformal invariance in dual space

⇒ Dual conformal covariance of scattering amplitudes!

Dual Superconformal symmetry

- Introduce dual on-shell superspace [Drummond, Henn, Korchemsky, Sokatchev]

$$(x_i - x_{i+1})^{\alpha\dot{\alpha}} = \lambda_i^\alpha\,\tilde{\lambda}_i^{\dot{\alpha}} \qquad (\theta_i - \theta_{i+1})^{\alpha\,A} = \lambda_i^\alpha\,\eta_i^A$$

  Then $x_i^{\alpha\dot{\alpha}}$ and $\theta_i^{\alpha\,A}$ have standard transformation law under (dual) conformal transformations

- Dual superconformal algebra, $\{\,P, M, \bar{M}, K, D \oplus R \oplus Q, \bar{Q}, S, \bar{S}\,\}$, acting in dual on-shell superspace $(x^{\alpha\dot{\alpha}}, \theta^{\alpha A})$:

$$P_{\alpha\dot{\alpha}} = \sum_i \partial_{i\alpha\dot{\alpha}}\,, \qquad Q_{\alpha A} = \sum_i \partial_{i\alpha A}$$

$$K^{\alpha\dot{\alpha}} = \sum_i x_i^{\alpha\dot{\beta}}\,x_i^{\dot{\alpha}\beta}\,\frac{\partial}{\partial x_i^{\beta\dot{\beta}}} + x_i^{\dot{\alpha}\beta}\,\theta_i^{\alpha\,B}\,\frac{\partial}{\partial\theta_i^{\beta\,B}}$$

$$S_\alpha^A = \sum_i -\theta_{i\alpha}^B\theta_i^{\beta A}\partial_{i\beta B} + x_{i\alpha}{}^{\dot{\beta}}\theta_i^{\beta A}\partial_{\beta\dot{\beta}}$$

$$\vdots$$

Dual superconformal symmetry of scattering amplitudes

- Extend dual superconformal generators so that they commute with constraints

$$(x_i - x_{i+1})^{\alpha\dot\alpha} = \lambda_i^\alpha \, \tilde\lambda_i^{\dot\alpha} \qquad (\theta_i - \theta_{i+1})^{\alpha\, A} = \lambda_i^\alpha \, \eta_i^A$$

leads to

$$K^{\alpha\dot\alpha} = \sum_i x_i^{\alpha\dot\beta} \, x_i^{\dot\alpha\beta} \, \frac{\partial}{\partial x_i^{\beta\dot\beta}} + x_i^{\dot\alpha\beta} \, \theta_i^{\alpha\, B} \, \frac{\partial}{\partial \theta_i^{\beta\, B}}$$

$$+ x_{i\dot\alpha}{}^{\beta} \lambda_{i\alpha} \partial_{i\beta} + x_{i+1\,\alpha}{}^{\dot\beta} \tilde\lambda_{i\dot\alpha} \partial_{i\dot\beta} + \tilde\lambda_{i\dot\alpha} \theta^B_{i+1\,\alpha} \partial_{iB}$$

- Indeed: Trees are dual superconformal covariant:

$$K^{\alpha\dot\alpha} \, \mathbb{A}_n^{\text{tree}} = -\sum_{i=1}^n x_i^{\alpha\dot\alpha} \, \mathbb{A}_n^{\text{tree}} \qquad S^{\alpha A} \, \mathbb{A}_n^{\text{tree}} = -\sum_{i=1}^n \theta_i^{\alpha A} \, \mathbb{A}_n^{\text{tree}}$$

$\Rightarrow$ $\boxed{\tilde K = K + \sum_i x_i \text{ and } \tilde S = S + \sum_i \theta_i}$ annihilate the amplitude.

- Beyond tree-level: Dual superconformal symmetry broken by IR divergences. However, breaking is under control and proportional to $\Gamma_{\text{cusp}}(g)$ for MHV amplitudes. Conjecture: Dual superconformal 'anomaly' is the same for MHV and non-MHV amplitudes [Drummond,Henn,Korchemsky,Sokatchev '08]

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The natural question

Q: What algebraic structure emerges when one commutes conformal with dual conformal generators? [Drummond,Henn,Plefka]

First Task: Tranform dual superconformal generators expressed in dual space (x_i, θ_i) into original on-shell superspace $(\lambda_i, \tilde{\lambda}_i, \eta_i)$!

1. Open chain by droping $x_{n+1} = x_1$ and $\theta_{n+1} = \theta_1$ conditions, implemented via δ-fcts: $\delta^{(4)}(p)\,\delta^{(8)}(q) = \delta^{(4)}(x_1 - x_{n+1})\,\delta^{(8)}(\theta_1 - \theta_{n+1})$
2. Express dual variables via "non-local' relations:

$$x_i^{\alpha\dot{\alpha}} = x_1^{\alpha\dot{\alpha}} + \sum_{j<i} \lambda_j^{\alpha}\,\tilde{\lambda}_j^{\dot{\alpha}} \qquad \theta_i^{\alpha A} = \theta_1^{\alpha A} + \sum_{j<i} \lambda_j^{\alpha}\,\eta_j^{A}$$

   Now set $x_1 = \theta_1 = 0$ by dual translation P and Poincare Susy Q.
3. Can now drop all x_1 and θ_i derivatives in dual superconformal generators.

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Dual $\mathfrak{psu}(2,2|4)$ generators

- Dual superconformal generators acting in standard on-shell superspace $(\lambda, \tilde{\lambda}, \eta)$:

$$P_{\alpha\dot{\alpha}} = 0\,, \qquad Q_{\alpha A} = 0\,, \qquad \overline{Q}^A_{\dot{\alpha}} = \sum_i \eta_i^A \partial_{i\dot{\alpha}} = \bar{s}^A_{\dot{\alpha}}$$

$$M_{\alpha\beta} = \sum_i \lambda_{i(\alpha}\partial_{i\beta)} = \bar{m}_{\dot{\alpha}\dot{\beta}}\,, \qquad \overline{M}_{\dot{\alpha}\dot{\beta}} = \sum_i \tilde{\lambda}_{i(\dot{\alpha}}\partial_{i\dot{\beta})} = m_{\alpha\beta}\,,$$

$$R^A{}_B = \sum_i \eta_i^A\,\partial_{iB} - \tfrac{1}{4}\delta^A_B \eta_i^C \partial_{iC} = -r^A{}_B\,,$$

$$D = \sum_i -\tfrac{1}{2}\lambda_i^\alpha \partial_{i\alpha} - \tfrac{1}{2}\tilde{\lambda}_i^{\dot{\alpha}}\partial_{i\dot{\alpha}} = -d\,,$$

$$C = \sum_i -\tfrac{1}{2}\lambda_i^\alpha \partial_{i\alpha} + \tfrac{1}{2}\tilde{\lambda}_i^{\dot{\alpha}}\partial_{i\dot{\alpha}} + \tfrac{1}{2}\eta_i^A \partial_{iA} = 1 - c\,,$$

$$S^A_\alpha = \sum_i \lambda_{i\alpha}\theta_i^{\gamma A}\partial_{i\gamma} + x_{i+1\,\alpha}{}^{\dot{\beta}}\eta_i^A \partial_{i\dot{\beta}} - \theta^B_{i+1\,\alpha}\eta_i^A \partial_{iB}\,,$$

$$\overline{S}_{\dot{\alpha}A} = \sum_i \tilde{\lambda}_{i\dot{\alpha}}\partial_{iA} = \bar{q}_{\dot{\alpha}A}\,,$$

$$K_{\alpha\dot{\alpha}} = \sum_i x_{i\dot{\alpha}}{}^{\beta}\lambda_{i\alpha}\partial_{i\beta} + x_{i+1\,\alpha}{}^{\dot{\beta}}\tilde{\lambda}_{i\dot{\alpha}}\partial_{i\dot{\beta}} + \tilde{\lambda}_{i\dot{\alpha}}\theta^B_{i+1\,\alpha}\partial_{iB}$$

Nonlocal structure of dual K and S

- We are left with the dual generators K and S, all others trivially related to standard superconformal generators.

$$\tilde{K}^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} x_i^{\dot{\alpha}\beta}\,\lambda_i^{\alpha}\,\frac{\partial}{\partial\lambda_i^{\beta}} + x_{i+1}^{\alpha\dot{\beta}}\,\tilde{\lambda}_i^{\dot{\alpha}}\,\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{\beta}}} + \tilde{\lambda}_i^{\dot{\alpha}}\,\theta_{i+1}^{\alpha B}\,\frac{\partial}{\partial\eta_i^{B}} + x_i^{\alpha\dot{\alpha}}$$

$$x_i^{\alpha\dot{\alpha}} = \sum_{j=1}^{i-1} \lambda_j^{\alpha}\,\tilde{\lambda}_j^{\dot{\alpha}} \qquad \theta_{i+1}^{\alpha A} = \sum_{j=1}^{i} \lambda_j^{\alpha}\,\eta_j^{A}$$

Nonlocal structure!

Yangian symmetry of scattering amplitudes in $\mathcal{N}=4$ SYM

- Can show that dual superconformal generators K and S may be lifted to level 1 generators of a Yangian algebra $Y[\mathfrak{psu}(2,2|4)]$:

$$[J_a^{(0)}, J_b^{(0)}\} = f_{ab}{}^c\, J_c^{(0)} \qquad \text{conventional superconformal symmetry}$$

$$[J_a^{(1)}, J_b^{(0)}\} = f_{ab}{}^c\, J_c^{(1)} \qquad \text{from dual conformal symmetry}$$

with nonlocal generators

$$\boxed{J_a^{(1)} = f^{cb}{}_a \sum_{1<j<i<n} J_{i,b}^{(0)}\, J_{j,c}^{(0)}}$$

and super Serre relations (representation dependent). [Dolan,Nappi,Witten]

$$[J_a^{(1)}, [J_b^{(1)}, J_c^{(0)}\}\} + (-1)^{|a|(|b|+|c|)}[J_b^{(1)}, [J_c^{(1)}, J_a^{(0)}\}\} + (-1)^{|c|(|a|+|b|)}[J_c^{(1)}, [J_a^{(1)}, J_b^{(0)}\}\}$$
$$= h(-1)^{|r||m|+|t||n|}\{J_l^{(0)}, J_m^{(0)}, J_n^{(0)}]\, f_{ar}{}^l f_{bs}{}^m f_{ct}{}^n f^{rst}.$$

Yangian symmetry of scattering amplitudes in $\mathcal{N}=4$ SYM

- Bosonic invariance $\boxed{p^{(1)}_{\alpha\dot\alpha}\,\mathbb{A}_n = 0}$ with

$$p^{(1)}_{\alpha\dot\alpha} = \tilde{K}_{\alpha\dot\alpha} + \Delta K_{\alpha\dot\alpha} = \frac{1}{2}\sum_{i<j}(m_{i,\alpha}{}^{\gamma}\delta^{\dot\gamma}_{\dot\alpha} + \bar{m}_{i,\dot\alpha}{}^{\dot\gamma}\delta^{\gamma}_{\alpha} - d_i\,\delta^{\gamma}_{\alpha}\delta^{\dot\gamma}_{\dot\alpha})\,p_{j,\gamma\dot\gamma} + \bar{q}_{i,\dot\alpha C}\,q^{C}_{j,\alpha} - (i\leftrightarrow j)$$

- In supermatrix notation: $\bar{A} = (\alpha, \dot\alpha | A)$

$$J^{\bar{A}}{}_{\bar{B}} = \begin{pmatrix} m^{\alpha}{}_{\beta} - \frac{1}{2}\,\delta^{\alpha}_{\beta}\,(d + \frac{1}{2}c) & k^{\alpha}{}_{\dot\beta} & s^{\alpha}{}_{B} \\ p^{\dot\alpha}{}_{\beta} & \overline{m}^{\dot\alpha}{}_{\dot\beta} + \frac{1}{2}\,\delta^{\dot\alpha}_{\dot\beta}\,(d - \frac{1}{2}c) & \bar{q}^{\dot\alpha}{}_{B} \\ q^{A}{}_{\beta} & \bar{s}^{A}{}_{\dot\beta} & -r^{A}{}_{B} - \frac{1}{4}\delta^{A}_{B}\,c \end{pmatrix}$$

and

$$\boxed{J^{(1)\,\bar{A}}{}_{\bar{B}} := -\sum_{i>j}(-1)^{|\bar{C}|}(J_i^{\bar{A}}{}_{\bar{C}}\,J_j^{\bar{C}}{}_{\bar{B}} - J_j^{\bar{A}}{}_{\bar{C}}\,J_i^{\bar{C}}{}_{\bar{B}})}$$

- Integrable spin chain picture also for colour ordered scattering amplitudes!
- Implies an infinite-dimensional symmetry algebra for $\mathcal{N}=4$ SYM scattering amplitudes!

Cyclicity?

- Potential problem [Beisert;Witten]: We have singled out particle 1 $\Leftrightarrow$ Yangian-generators are not cyclic **but** color ordered scattering amplitudes are cyclic??
- Resolution: Consider the Yangian generators produced by singling out particle 2:

$$\tilde{J}^{(1)}_a = f^{cb}{}_a \sum_{2<j<i<n+1} J^{(0)}_{i,b}\, J^{(0)}_{j,c}$$

then one shows

$$J^{(1)\,\bar{A}}{}_{\bar{B}} - \tilde{J}^{(1)\,\bar{A}}{}_{\bar{B}} = \delta^{\bar{A}}_{\bar{B}}\, J_1^{\bar{C}}{}_{\bar{C}} = \ldots = \delta^{\bar{A}}_{\bar{B}}\, c_1$$

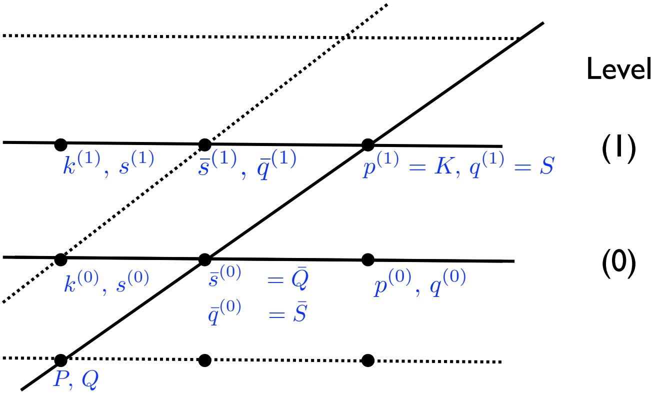
Importantly $c_i\, \mathbb{A}_n = 0$ locally! Hence level one generators $J^{(1)\,\bar{A}}{}_{\bar{B}}$ are cyclic when acting on amplitudes.

- Linked to vanishing Killing form of superalgebra $(-1)^{|c|} f_{ac}{}^d\, f_{bd}{}^c = 0$

$\Longrightarrow$ [K. Zarembo's talk]

Summary

- Combination of standard and dual superconformal symmetry lifts to Yangian $Y[\mathfrak{psu}(2,2|4)]$ [Picture: Beisert]



- Tree level superamplitudes invariant: $\boxed{\mathcal{J} \circ \mathbb{A}_n^{\text{tree}} = 0}$ for $\mathcal{J} \in Y[\mathfrak{psu}(2,2|4)]$.

Outlook

- Same Yangian symmetry appears in the spectral problem of AdS/CFT!

  [Dolan,Nappi,Witten;Beisert, Zwiebel, Torrieli, de Leeuw,...]

  $\Rightarrow$ Strong hint for integrability in scattering amplitudes!

- Challenge at weak coupling: Does Yangian symmetry extend to the loop level?

  Breaking of dual conformal invariance at loop level under control [Korchemsky's talk]
  Can breaking of standard conformal invariance at loop level be controlled?

- Amplitudes at Strong coupling: Invariant under Yangian symmetry?
- Is form of tree amplitudes fixed by Yangian symmetry?

  $\Rightarrow$ Needs to include colinear limits $\equiv$ length changing effects

  [Korchemsky's and McLoughlin's talk]

- Can it constrain the form of the higher loop amplitudes?

  In particular is the 'remainder' function for $n \geq 6$ point MHV amplitudes fixed?