Quantization of Integrable Systems and Supersymmetric Vacua

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$k \to \infty$ limit of Gauged WZW_k (Verlinde formula).

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$\mathbf{\Sigma}$ - the twisted chiral multiplets: adjoint complex scalar, gauge field strength: $\mathbf{\Sigma} = \mathcal{D}_+ \bar{\mathcal{D}}_- V$.

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$$D: \qquad \int \mathrm{d}^2x\,\mathrm{d}^4\theta\,\left(-\frac{1}{4\mathrm{e}^2}\,\mathrm{tr}\,\boldsymbol{\Sigma}\bar{\boldsymbol{\Sigma}} + K(e^{\mathbf{V}/2}\,\mathbf{X}\;,\;\bar{\mathbf{X}}\,e^{\mathbf{V}/2})\right)$$

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Some m_i break $\mathcal{N}=4$ to $\mathcal{N}=2$ (iu), some don't (μ). One can not turn on these unless there is some global symmetry unbroken.

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In addition there are other massive fields which can be integrated out on the Coulomb branch. These are the $\mathbf{g}/\mathbf{t}$-components ($\mathbf{g}$ - Lie algebra corresponding to Lie group G, $\mathbf{t}$ - its Cartan sub-algebra) of the vector multiplets, the W-bosons ...

Of all terms in effective Lagrangian the twisted F-terms, i.e. the effective twisted superpotential $\tilde{W}^{\mathrm{eff}}(\Sigma) \Leftrightarrow \tilde{W}^{\mathrm{eff}}(\sigma + ...)$, can be computed exactly - receives only one-loop contributions.
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Only $d\tilde{W}^{\mathrm{eff}}(\sigma)$ enters in effective Lagrangian $\mathcal{L}$.

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Again, one can write the explicit formula for effective twisted superpotential in 2d.

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Than true vacuum state will be a particular, "harmonic", representative in this cohomology.

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SUSY vacua form the representation of chiral ring. Basically, for every $\mathcal{N}=2$ theory there is a quantum integrable system (assuming all good conditions like discrete specturm of vacua etc.)

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SUSY vacua - we need to minimize the potential on Coulomb branch.

Suppose we have the theory with the effective twisted superpotential $\tilde{W}^{\mathrm{eff}}(\sigma); \sigma = \left(\sigma^i\right)_{i=1}^{r}$ parameterize the Coulomb branch (the complexification of the Lie algebra $\mathbf{t} = \mathrm{Lie}\mathbf{T}$ of the unbroken gauge group $\mathbf{T}$, which is abelian).

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$$U_{\vec{n}} = \frac{1}{2}\mathrm{g}^{ij}\left(-2\pi i n_i + \frac{\partial \tilde{W}^{\mathrm{eff}}}{\partial \sigma^i}\right)\left(+2\pi i n_j + \frac{\partial \bar{\tilde{W}}^{\mathrm{eff}}}{\partial \bar{\sigma}^j}\right)$$

$$\frac{1}{2\pi i}\frac{\partial \tilde{W}^{\mathrm{eff}}(\sigma)}{\partial \sigma^i} = n_i; \qquad \exp\left(\frac{\partial \tilde{W}^{\mathrm{eff}}(\sigma)}{\partial \sigma^i}\right) = 1$$

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The adjoint $\mathbf{\Phi}$ is a part of the vector multiplet in $4d$, while chiral fundamental and anti-fundamentals combine into hypermultiplet in the fundamental representation. We are dealing, therefore, with the matter content of the four dimensional $\mathcal{N} = 2$ theory with $N_c = N$, $N_f = L$.

Since the gauge group has a center $U(1)$ one can turn on the Fayet-Illiopoulos term, and the theta angle, which we combine into a complexified coupling $\theta \mapsto t = \frac{\theta}{2\pi} + ir$.

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$$\tilde{W}_{\tilde{Q}\Phi Q} = \sum_{i=1}^{N}\sum_{a=1}^{L}[\left(\sigma_i + m_a^{\mathrm{f}}\right)\left(\log\left(\sigma_i + m_a^{\mathrm{f}}\right) - 1\right) +$$

$$+ \left(-\sigma_i + m_a^{\bar{\mathrm{f}}}\right)\left(\log\left(-\sigma_i + m_a^{\bar{\mathrm{f}}}\right) - 1\right)] +$$

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where $\mu_a \in \mathbf{C}$, $u \in \mathbf{C}$, s_a-half-integer.

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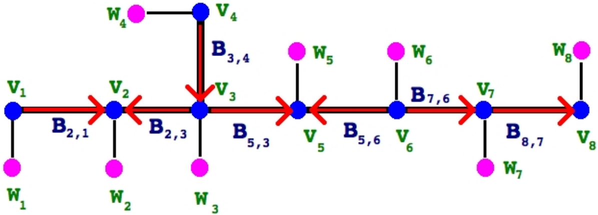

$$\mathcal{W} = \sum_{\langle \mathbf{i},\mathbf{j}\rangle} tr_{V_{\mathbf{i}}}\left(B_{\mathbf{i},\mathbf{j}}\Phi_{\mathbf{j}}\tilde{B}_{\mathbf{j},\mathbf{i}}\right) - tr_{V_{\mathbf{j}}}\left(\tilde{B}_{\mathbf{j},\mathbf{i}}\Phi_{\mathbf{i}}B_{\mathbf{i},\mathbf{j}}\right) +$$

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$C_{\mathbf{ij}}$ is the Cartan matrix of the corresponding A_r, D_r, E_r type:
$C_{\mathbf{ij}} = \langle \alpha_{\mathbf{i}}, \alpha_{\mathbf{j}} \rangle$, $\alpha_{\mathbf{i}}$ - simple roots of the corresponding Lie algebra.

$$\mathcal{W} = \sum_{\langle \mathbf{i},\mathbf{j}\rangle} tr_{V_\mathbf{i}}\left(B_{\mathbf{i},\mathbf{j}}\Phi_\mathbf{j}\tilde{B}_{\mathbf{j},\mathbf{i}}\right) - tr_{V_\mathbf{j}}\left(\tilde{B}_{\mathbf{j},\mathbf{i}}\Phi_\mathbf{i}B_{\mathbf{i},\mathbf{j}}\right) +$$

$$\sum_{\mathbf{i}=1}^{r}\sum_{a=1}^{L_\mathbf{i}} \varpi_{\mathbf{i},\mathbf{a}}\left(Q_\mathbf{i}^a \Phi_\mathbf{i}^{2s_{\mathbf{i},\mathbf{a}}}\tilde{Q}_{\mathbf{i},\mathbf{a}}\right)$$

It is convenient to denote the fields $\tilde{B}, B, \Phi$ collectively by $\mathbf{X}$.
Global symmetry - $\mathbf{T} = U(1) \times \times_{\mathbf{i}=1}^{r} U(1)^{L_\mathbf{i}}$

$$\mathbf{X}_{\mathbf{i},\mathbf{j}} \mapsto e^{-\frac{1}{2}C_{\mathbf{ij}}u}\mathbf{X}_{\mathbf{i},\mathbf{j}}; Q_{\mathbf{i},\mathbf{a}} \mapsto e^{i\mu_{\mathbf{i},\mathbf{a}} - is_{\mathbf{i},\mathbf{a}}u}Q_{\mathbf{i},\mathbf{a}}; \tilde{Q}_{\mathbf{i},\mathbf{a}} \mapsto e^{-i\mu_{\mathbf{i},\mathbf{a}} - is_{\mathbf{i},\mathbf{a}}u}\tilde{Q}_{\mathbf{i},\mathbf{a}}$$

$C_{\mathbf{ij}}$ is the Cartan matrix of the corresponding A_r, D_r, E_r type:
$C_{\mathbf{ij}} = \langle \alpha_\mathbf{i}, \alpha_\mathbf{j}\rangle$, $\alpha_\mathbf{i}$ - simple roots of the corresponding Lie algebra.

Thus we can turn on twisted masses: $\mu_{\mathbf{i},\mathbf{a}}; u$.

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Can be lifted to $3d$ and $4d$ - trigonometric and elliptic equations.

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Algebraic completely integrable system (**ACIS**)- complexification: $T^*(C^\times)^N$ with holomorphic symplectic form $\Omega^{2,0} = \sum dp_i \wedge dq_i$.

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$\mathcal{F}(a)$ - prepotential for pure $\mathcal{N}=2$ theory in 4d, defines low energy effective action in pure $\mathcal{N}=2$ theory.

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$$\mathcal{P}(x) = \sum_{n \in Z} \frac{1}{sinh^2(x + n\beta)} = u_0(x) + \sum_{k=1}^{\infty} q^k u_n(x)$$

$$q = e^{-2\beta}; \quad u_0 = \frac{1}{sinh^2 x} = \sum_k n e^{-kx}; \quad u_k(x) = 4 \sum_{d|k} d(e^{dx} + e^{-dx})$$

Similar to Periodic Toda - for eCM there is Lax operator:

$$\Phi_{ij}(p,q;z) = p_i\delta_{ij} + m\frac{\Theta(z+q_i-q_j)\Theta'(0)}{\Theta(q_i-q_j)\Theta(z)}(1-\delta_{ij})$$

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$$< \phi >= diag(a_i, ..., a_N), \quad \mathcal{F}(a) = \mathcal{F}^{pert}(a) + \mathcal{F}^{non-pert}(a)$$

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$$\mathcal{F}^{pert}(a;\tau,m) = \frac{\tau}{2}\sum_{i=1}^{N} a_i^2 + \frac{3N^2m^2}{2} + \frac{1}{4}\sum_{i,j=1}^{N}[(a_i-a_j)^2 log(a_i-a_j) - \\ -(a_i-a_j+m)^2 log(a_i-a_j+m)]$$

Similar to Periodic Toda - for eCM there is Lax operator:

$$\Phi_{ij}(p,q;z) = p_i\delta_{ij} + m\frac{\Theta(z+q_i-q_j)\Theta'(0)}{\Theta(q_i-q_j)\Theta(z)}(1-\delta_{ij})$$

Θ - odd theta function on the elliptic curve with $\tau = \frac{i\beta}{\pi}$.

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Our main examples - pToda (pure $\mathcal{N}=2$) and eCM ($\mathcal{N}=2^*$).

$\mathcal{N}=2$ gauge theory on $R^2 \times R^2_\epsilon$ is a deformation of $\mathcal{N}=2$ theory on $R^2 \times R^2$ with one, equivariant, parameter ϵ which corresponds to the rotation of second R^2 around its origin. Denote corresponding vector field $V = \epsilon(x^2\partial_3 - x^3\partial_2)$. Bosonic part is:

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This theory has twisted/topological formulation (together with usual deformation - $t_1, ..., t_N$, as in $LNS \quad '97$), ϵ deformation of Donaldson-Witten, and its abelianization (effective low energy) is 2d gauge theory with four Q's and superpotential (for $\mathcal{N}=2^*$):

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$W = W_{pert}(a|t_1, ..., t_N; m, \epsilon, \tau) + \sum_{k=1}^{\infty} q^k W_k(a|t_1, ..., t_N; m, \epsilon)$ is known exactly.

What is exactly the quantization problem for which this $W(a|t_1, ..., t_N; m, \epsilon, \tau)$ gives the Yang function and thus - the exact spectrum?

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For pToda we replace phase space coordinate $p_i = \epsilon \frac{\partial}{\partial q_i}$, write the eigenvalue problem for all Hamiltonians, parametrize eigenvalues $E_1, ..., E_N$ in terms of $a_1, ..., a_N$ - e. g. for H_2:

$$[\epsilon^2 \sum_{i=1}^{N} \frac{\partial^2}{\partial {q_i}^2} + \Lambda^2 (\sum_{i=1}^{N-1} e^{q_i - q_{i-1}} + e^{q_N - q_1})]\Psi(q) = E_2(a)\Psi(q)$$

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For pToda and eCM these claims are checked in Λ^2 and q-expansion knowing W from SYM exactly for $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$.